UNIQUENESS OF THE FIXED POINT OF NONEXPANSIVE SEMIDIFFERENTIABLE MAPS

MARIANNE AKIAN, STÉPHANE GAUBERT, AND ROGER NUSSBAUM

ABSTRACT. We consider semidifferentiable (possibly nonsmooth) maps, acting on a subset of a Banach space, that are nonexpansive either in the norm of the space or in the Hilbert's or Thompson's metric inherited from a convex cone. We show that the global uniqueness of the fixed point of the map, as well as the geometric convergence of every orbit to this fixed point, can be inferred from the semidifferential of the map at this point. In particular, we show that the geometric convergence rate of the orbits to the fixed point can be bounded in terms of Bonsall's non-linear spectral radius of the semidifferential. We derive similar results concerning the uniqueness of the eigenline and the geometric convergence of the orbits to it, in the case of positively homogeneous maps acting on the interior of a cone, or of additively homogeneous maps acting on an AM-space with unit. This is motivated in particular by the analysis of dynamic programming operators (Shapley operators) of zero-sum stochastic games.

1. INTRODUCTION

Non-linear maps acting on a subset of a Banach space, that are nonexpansive either in the norm of the space, or in metrics inherited from a convex cone, like Hilbert's or Thompson's metric, arise in a number of fields, including population dynamics [Per07], entropy maximization and scaling problems [MS69, BLN94], renormalization operators and fractal diffusions [Sab97, Met05], mathematical economy [Mor64], optimal filtering [Bou95], optimal control [CT80, AG03] and zero-sum games [Kol92, FV97, RS01, Ney03, AGG12]. In particular, the dynamic programming operators, known as Bellman operators in control theory, or as Shapley operators in game theory, turn out to be, under standard assumptions, sup-norm nonexpansive maps defined on a space of continuous functions.

Such nonexpansive maps include as a special case the nonnegative matrices and positive linear operators arising in Perron-Frobenius theory. A number of works, including [KR48, Bir57, Bir62, Hop63, Kra64, Bir67, Pot77, Bus73, Bus86, Nus88, Nus89, Kra01, NVL99, GG04, AGLN06, GV12, LN12], have dealt with the extension of Perron-Frobenius theory to the non-linear case.

Date: October 22, 2012.

Key words and phrases. Nonlinear eigenvector, Hilbert's metric, Thompson's metric, nonexpansive maps, AM-space with unit, semidifferentiability, non-linear spectral radius, geometric convergence, zero-sum stochastic games, value iteration.

The first two authors were partially supported by the Arpege programme of the French National Agency of Research (ANR), project "ASOPT", number ANR-08-SEGI-005.

The third author was partially supported by NSFDMS 0701171 and by NSFDMS 1201328.

In particular, a central problem is to give conditions allowing one to check whether a given fixed point is globally unique, and whether all the orbits of the maps converge to it.

This problem was considered by Nussbaum in [Nus88], who gave general conditions in the case of differentiable maps. Several results in [Nus88], show, in various settings, that a given fixed point of the map is globally unique as soon as the derivative of the map at this fixed point does not have any non-trivial fixed point, provided some mild compactness condition is satisfied. Similarly, the global geometric convergence of all the orbits of the map to this fixed point is guaranteed if the spectral radius of the same derivative is strictly less than one. Analogous results are derived in [Nus88] for non-linear eigenvectors.

In a number of applications, including the ones arising from control and games, differentiability assumptions turn out to be too restrictive, since Bellman or Shapley operators are typically nonsmooth (smoothness being related to the uniqueness of the optimal action). However, such operators often satisfy a weaker condition, *semidifferentiability*, which was first introduced by Penot [Pen82], and which has become a basic notion in variational analysis, see in particular the book by Rockafellar and Wets [RW98].

The case of nondifferentiable convex Shapley operators (corresponding to one player stochastic games) was studied by Akian and Gaubert in [AG03]. It was shown there that the dimension of the fixed point set can be bounded by considering the subdifferential of the operator at any fixed point (in particular, the uniqueness of this fixed point, perhaps up to an additive constant, can be guaranteed by considering this subdifferential), and that the asymptotic behavior of the orbits, could also be inferred from this subdifferential.

In this paper, we show that the general principle, developed in [Nus88] in the differentiable case and in [AG03] in the convex nondifferentiable case carries over to the nonconvex semidifferentiable case: the global uniqueness of the fixed point of a nonexpansive map, and the geometric convergence of the orbits to it, can be guaranteed by considering only the infinitesimal behavior of the map at the fixed point.

Our main results are Theorem 6.1, dealing with the uniqueness of the fixed point, and Theorem 6.8, dealing with the geometric convergence of the orbits. We also derive in §7 similar results for the uniqueness of the normalized eigenvector, and for the convergence of the orbits, in the case of homogeneous maps acting on the interior of a normal cone and nonexpansive in Hilbert's or Thompson's metric. Note that the spectral radius arising in the differentiable case is replaced by Bonsall's cone spectral radius in the semidifferentiable case. Moreover, to state the uniqueness result, a non-linear Fredholm-type compactness is required. In this way, we generalize to the case of semidifferentiable maps (Theorem 7.5, and Corollaries 7.7 and 10.5) some of the main results of [Nus88], including Theorem 2.5 there. Note that the present results are already new in the finite dimensional case. Then, the technical compactness conditions are trivially satisfied.

We remark in passing that there are interesting classes of cone maps $f : C \to C$ which are differentiable on the interior of the cone C but only semi-differentiable on the boundary of C. See, for example, the class $\mathcal{M}_-$ treated in [Nus89]. Properties equivalent to semidifferentiability (see e.g. Theorem 3.1 there) have been used

there to prove existence of eigenvectors in the interior of C for some such maps f. However, we shall not pursue this line of applications here.

As an illustration of our results, we analyze a simple zero-sum stochastic repeated game (§10.2). (The application to zero-sum games will be discussed in more detail in a further work.) We note in this respect that the spectral radius of the semidifferential of a Shapley operator can be computed using the results of the companion work [AGN11], which yields explicit formulæ for the geometric contraction rate.

The paper is organized as follows. Some basic definitions and results concerning convex cones, Hilbert's and Thompson's metric, are recalled in §2. Preliminary results concerning semidifferentiable maps are established in §3. Then, in §4, we examine the semidifferentials of order preserving or nonexpansive maps. The needed results concerning the different notions of nonlinear spectral radius, and in particular Bonsall's cone spectral radius, are recalled in §5 (these results are essentially taken from [MPN02] and [AGN11]). Then, the main results of the paper, concerning the uniqueness of the fixed point, and the geometric convergence of the orbits, are established in §6. The corollaries of these results concerning non-linear eigenvectors of homogeneous maps leaving invariant the interior of a cone are derived in §7. Note that the proofs in §6 and 7 are based on metric fixed point technique (approximation of the map by a strictly contracting map). In particular, in the case of Hilbert's or Thompson's metric, the need to work with a complete metric space limits the scope of these results to *normal* cones. Hence, alternative uniqueness results, using degree theory arguments, which are valid more generally in proper (closed, convex, and pointed) cones, are presented in §8. Then, in §9, the results are specialized to differentiable maps, and compared with the ones of [Nus88]. Finally, in §10, the present results are adapted to the additive setting (order preserving and additively homogeneous maps acting on an AM-space with unit, or equivalently, on a space of continuous functions $\mathscr{C}(K)$ for some compact set K), and illustrated by the analysis of an example of zero-sum stochastic game (a variant of the Richman [LLP+99] or stochastic tug-of-war games [PSSW09]).

2. Preliminary results about Hilbert's and Thompson's metric

In this section, we recall classical notions about cones, and recall or establish results which will be used in the following sections. See [Nus88, Chapter 1] and [Nus94, Section 1] for more background.

2.1. **Hilbert's and Thompson's metric.** In this paper, a subset C of a real vector space X is called a *cone* (with vertex 0) if $tC := \{tx \mid x \in C\} \subset C$ for all $t \geqslant 0$. If f is a map from a cone C of a vector space X to a cone C' of a vector space Y, we shall say that f is (positively) *homogeneous* (of degree 1) if $f(ty) = tf(y)$, for all $t > 0$ and $y \in C$. We say that the cone C is *pointed* if $C \cap (-C) = \{0\}$. A convex pointed cone C of X induces on X a partial ordering $\leqslant_C$, which is defined by $x \leqslant_C y$ iff $y - x \in C$. If C is obvious, we shall write $\leqslant$ instead of $\leqslant_C$. When X is a topological vector space, we say that C is *proper* if it is closed convex and pointed. Note that in [Nus88], a *cone* is by definition what we call here a proper cone. We next recall the definition of Hilbert's and Thompson's metrics associated to C.

Let $x \in C \setminus \{0\}$ and $y \in X$. We define $M(y\,/\,x)$ by

$$M(y\,/\,x) := \inf\{b \in \mathbb{R} \mid y \leqslant bx\} \enspace , \tag{2.1}$$

where the infimum of the emptyset is by definition equal to $+\infty$. Similarly, we define $m(y\,/\,x)$ by

$$m(y\,/\,x) := \sup\{a \in \mathbb{R} \mid ax \leqslant y\} \enspace , \tag{2.2}$$

where the supremum of the emptyset is by definition equal to $-\infty$. We have $m(y\,/\,x) = -M(-y\,/\,x)$ and if in addition $y \in C \setminus \{0\}$, $m(y\,/\,x) = 1/M(x\,/\,y)$ (with $1/(+\infty) = 0$). Since C is pointed and closed, we have $M(y\,/\,x) \in \mathbb{R} \cup \{+\infty\}$, and $y \leqslant M(y\,/\,x)x$ as soon as $M(y\,/\,x) < +\infty$. Symmetrically $m(y/x) \in \mathbb{R} \cup \{-\infty\}$ and $m(y\,/\,x)x \leqslant y$, as soon as $m(y/x) > -\infty$.

We shall say that two elements x and y in C are *comparable* and write $x \sim y$ if there exist positive constants $a > 0$ and $b > 0$ such that $ax \leqslant y \leqslant bx$. If $x, y \in C \setminus \{0\}$ are comparable, we define

$$\begin{aligned} d(x, y) &= \log M(y\,/\,x) - \log m(y\,/\,x) \enspace , \\ \bar{d}(x, y) &= \log M(y\,/\,x) \vee - \log m(y\,/\,x) \enspace . \end{aligned}$$

We also define $d(0, 0) = \bar{d}(0, 0) = 0$.

If $u \in C$, we define

$$C_u = \{x \in C \mid x \sim u\} \enspace .$$

If C has nonempty interior $\operatorname{int} C$, and $u \in \operatorname{int} C$, then $C_u = \operatorname{int} C$. In general, $C_u \cup \{0\}$ is a pointed convex cone but $C_u \cup \{0\}$ is not closed. The map $\bar{d}$ is a metric on C_u, called *Thompson's metric*. The map d is called the *Hilbert's projective metric* on C_u. The term "projective metric" is justified by the following properties: for all $x, y, z \in C_u$, $d(x, z) \leqslant d(x, y) + d(y, z)$, $d(x, y) = d(y, x) \geqslant 0$ and $d(x, y) = 0$ iff $y = \lambda x$ for some $\lambda > 0$.

From now, we will assume that $X = (X, \|\cdot\|)$ is a Banach space. We denote by X^* the space of continuous linear forms over X, and by $C^* := \{\psi \in X^* \mid \psi(x) \geqslant 0 \ \forall x \in C\}$ the *dual cone* of C. If f is a map between two ordered sets $(D, \leqslant)$ and $(D', \leqslant)$, we shall say that f is *order preserving* if $f(x) \leqslant f(y)$ for all $x, y \in D$ such that $x \leqslant y$. Then, any element of C^* is a homogeneous and order preserving map from $(X, \leqslant_C)$ to $[0, +\infty)$. Since C is proper, the Hahn-Banach theorem implies that for all $u \in C \setminus \{0\}$, there exists $\psi \in C^*$ such that $\psi(u) > 0$. For such a ψ, we have $\psi(x) > 0$ for all $x \in C_u$. More generally, if $q : C_u \to (0, +\infty)$ is homogeneous and order preserving, we denote

$$\Sigma_u = \{x \in C_u \mid q(x) = q(u)\} \enspace . \tag{2.3}$$

Then, d and $\bar{d}$ are equivalent metrics on Σ_u. Indeed, as shown in [Nus88, Remark 1.3, p. 15]:

$$\frac{1}{2} d(x, y) \leqslant \bar{d}(x, y) \leqslant d(x, y),\ \forall x, y \in \Sigma_u \enspace . \tag{2.4}$$

More precisely:

$$1 \leqslant M(y\,/\,x) \leqslant e^{d(x,y)} \quad \forall x, y \in \Sigma_u \enspace . \tag{2.5}$$

To see this, let us apply q to the inequality $y \leqslant M(y\,/\,x)\, x$. Using that q is order preserving and homogeneous, and that $q(x) = q(y)$, we get $M(y\,/\,x) \geqslant 1$. By symmetry, $M(x\,/\,y) \geqslant 1$, hence $\log M(y\,/\,x) = d(x, y) - \log M(x\,/\,y) \leqslant d(x, y)$.

We say that a cone C is *normal* if C is proper and there exists a constant M such that $\|x\| \leqslant M\|y\|$ whenever $0 \leqslant x \leqslant y$. Every proper cone C in a finite dimensional Banach space $(X, \|\cdot\|)$ is necessarily normal. We shall need the following result of Thompson.

Proposition 2.1 ([Tho63, Lemma 3])**.** *Let C be a normal cone in a Banach space $(X, \|\cdot\|)$. For all $u \in C \setminus \{0\}$, $(C_u, \bar{d})$ is a complete metric space.*

The following result follows from a general result of Zabreĭko, Krasnosel′skiĭ and Pokornyĭ [ZKP71] (see [Nus88, Theorem 1.2 and Remarks 1.1 and 1.3] and a previous result of Birkhoff [Bir62]). When $q \in C^*$, it follows from Proposition 2.1, (2.4) and the property that Σ_u is closed in the topology of the Thompson's metric $\bar{d}$.

Proposition 2.2. *Let C be a normal cone in a Banach space $(X, \|\cdot\|)$. Let $u \in C \setminus \{0\}$ and $q : C_u \to (0, +\infty)$, which is homogeneous and order preserving with respect to C. Let $\Sigma_u = \{x \in C_u \mid q(x) = 1\}$. Then, (Σ_u, d) and $(\Sigma_u, \bar{d})$ are complete metric spaces.*

2.2. **The local Banach space X_u.** Given $u \in C \setminus \{0\}$, we define the linear space $X_u = \{x \in X \mid \exists a > 0,\ -au \leqslant x \leqslant au\}$. We will show (in Lemma 2.3 and Proposition 2.4 below) that X_u is equipped with a norm $\|\cdot\|_u$, and a seminorm ω_u, such that $\bar{d}$ and d behave locally, near u, as $\|\cdot\|_u$ and ω_u, respectively.

Let M and m be defined as in (2.1) and (2.2). We equip X_u with the norm:

$$\|x\|_u = M(x \,/\, u) \vee -m(x \,/\, u) = \inf\{a > 0 \mid -au \leqslant x \leqslant au\} \ . \tag{2.6}$$

We also define the *oscillation* of $x \in X_u$:

$$\omega_u(x) = M(x \,/\, u) - m(x \,/\, u) = \inf\{b - a \mid au \leqslant x \leqslant bu\} \ . \tag{2.7}$$

The map $\omega_u : x \mapsto \omega_u(x)$ is a seminorm on X_u, which satisfies $\omega_u(x) = 0 \iff x \in \mathbb{R}u$. Since $M(x \,/\, u) \vee -m(x \,/\, u) \geqslant \frac{1}{2}(M(x \,/\, u) - m(x \,/\, u))$, we get

$$\frac{1}{2}\omega_u(x) \leqslant \|x\|_u, \qquad \forall x \in X_u \ . \tag{2.8}$$

Moreover, when ψ is an element of C^* and $\psi(u) = 1$, $\psi(x) = 0$ and $x \leqslant au$ implies $0 \leqslant a$, so that $M(x \,/\, u) \geqslant 0$, and dually, $m(x \,/\, u) \leqslant 0$. Hence, $M(x \,/\, u) \vee -m(x \,/\, u) \leqslant M(x \,/\, u) - m(x \,/\, u)$, which gives

$$\|x\|_u \leqslant \omega_u(x) \quad \forall x \in X_u, \text{ such that } \psi(x) = 0 \ . \tag{2.9}$$

Gathering (2.8) and (2.9), we see that the restriction of ω_u to the subspace $\{x \in X_u \mid \psi(x) = 0\}$ of X_u is a norm equivalent to $\|\cdot\|_u$.

We shall need the following variants or refinements of some inequalities of [Nus94, Proposition 1.1].

Lemma 2.3. *Let C be a proper cone, and $u \in C \setminus \{0\}$. For all $x, y \in C_u$, we have:*

$$\|x - y\|_u \leqslant (e^{\bar{d}(x,y)} - 1)e^{\bar{d}(x,u) \wedge \bar{d}(y,u)} \ , \tag{2.10}$$

$$\bar{d}(x, y) \leqslant \log(1 + \|x - y\|_u \, e^{\bar{d}(x,u) \vee \bar{d}(y,u)}) \ . \tag{2.11}$$

Moreover, defining Σ_u as in (2.3), with $q := \psi$, we get that for all $x, y \in \Sigma_u$,

$$\omega_u(x - y) \leqslant (e^{d(x,y)} - 1)e^{\bar{d}(x,u) \wedge \bar{d}(y,u)} \ , \tag{2.12}$$

$$d(x, y) \leqslant \omega_u(x - y)e^{\bar{d}(x,u) \vee \bar{d}(y,u)} \ . \tag{2.13}$$

Proof. We first prove (2.10). By definition of $\bar{d}$, we can write

$$e^{-\bar{d}(y,u)}u \leqslant y \leqslant e^{\bar{d}(y,u)}u \tag{2.14}$$

$$e^{-\bar{d}(x,y)}y \leqslant x \leqslant e^{\bar{d}(x,y)}y \ . \tag{2.15}$$

Using (2.15) and the second inequality in (2.14), we get

$$\begin{aligned}(e^{-\bar{d}(x,y)}-1)e^{\bar{d}(y,u)}u &\leqslant (e^{-\bar{d}(x,y)}-1)y \\ &\leqslant x-y \leqslant (e^{\bar{d}(x,y)}-1)y \leqslant (e^{\bar{d}(x,y)}-1)e^{\bar{d}(y,u)}u \ .\end{aligned}$$

Hence,

$$\|x-y\|_u \leqslant \big((e^{\bar{d}(x,y)}-1)\vee(1-e^{-\bar{d}(x,y)})\big)e^{\bar{d}(y,u)} = (e^{\bar{d}(x,y)}-1)e^{\bar{d}(y,u)} \ .$$

Together with the symmetrical inequality obtained by exchanging x and y, this yields (2.10).

We next prove (2.11). By definition of $\|\cdot\|_u$, we can write

$$x-y \leqslant \|x-y\|_u u$$

for all $x,y \in X_u$. If $x,y\in C_u$, using the first inequality in (2.14), we deduce

$$x \leqslant y + \|x-y\|_u u \leqslant (1+\|x-y\|_u e^{\bar{d}(y,u)})y \ ,$$

and by symmetry,

$$y \leqslant (1+\|x-y\|_u e^{\bar{d}(x,u)})x \ .$$

It follows that

$$\begin{aligned}\bar{d}(x,y) &\leqslant \log(1+\|x-y\|_u e^{\bar{d}(y,u)}) \vee \log(1+\|x-y\|_u e^{\bar{d}(x,u)}) \\ &= \log(1+\|x-y\|_u e^{\bar{d}(y,u)\vee\bar{d}(x,u)}) \ ,\end{aligned}$$

which shows (2.11).

We now prove (2.12). By definition of d, we can write

$$d(x,y) = \log\beta - \log\alpha, \text{ with } \alpha,\beta>0 \text{ and} \tag{2.16}$$

$$\alpha y \leqslant x \leqslant \beta y \ . \tag{2.17}$$

When $x,y\in\Sigma_u$, applying ψ to (2.17), we get $\alpha\leqslant 1\leqslant\beta$. Combining the second inequality in (2.17) with $\beta-1\geqslant 0$ and the second inequality in (2.14), we get

$$x-y \leqslant (\beta-1)y \leqslant (\beta-1)e^{\bar{d}(y,u)}u \ . \tag{2.18}$$

Combining the first inequality in (2.17) with $\alpha-1\leqslant 0$ and the second inequality in (2.14), we get

$$x-y \geqslant (\alpha-1)y \geqslant (\alpha-1)e^{\bar{d}(y,u)}u \ . \tag{2.19}$$

Gathering (2.18) and (2.19), we get

$$\omega_u(x-y) \leqslant (\beta-\alpha)e^{\bar{d}(y,u)} \ .$$

By symmetry, $\omega_u(x-y)\leqslant(\beta-\alpha)e^{\bar{d}(x,u)}$, so that

$$\omega_u(x-y) \leqslant (\beta-\alpha)e^{\bar{d}(x,u)\wedge\bar{d}(y,u)} \ .$$

Since $\beta\geqslant 1\geqslant\alpha$, we get

$$\omega_u(x-y) \leqslant (\frac{\beta}{\alpha}-1)e^{\bar{d}(x,u)\wedge\bar{d}(y,u)} = (e^{d(x,y)}-1)e^{\bar{d}(x,u)\wedge\bar{d}(y,u)} \ ,$$

which shows (2.12).

We finally prove (2.13). By definition of ω_u, we have

$$au \leqslant x - y \leqslant bu \tag{2.20}$$

for some $a, b \in \mathbb{R}$, with $b - a = \omega_u(x - y)$. Applying ψ to (2.20), we get $a \leqslant 0 \leqslant b$. Combining the second inequality in (2.20) with the first inequality in (2.14), we get

$$x \leqslant y + bu \leqslant (1 + be^{\bar{d}(y,u)})y \tag{2.21}$$

and by symmetry ((2.20) is equivalent to $-bu \leqslant y - x \leqslant -au$)

$$y \leqslant x - au \leqslant (1 - ae^{\bar{d}(x,u)})x \tag{2.22}$$

Gathering (2.21) and (2.22), we get

$$d(x, y) \leqslant \log(1 + be^{\bar{d}(y,u)}) + \log(1 - ae^{\bar{d}(x,u)}) \ .$$

Since $\log(1 + x) \leqslant x$ for all $x \geqslant 0$ and $b \geqslant 0 \geqslant a$, we get

$$d(x, y) \leqslant be^{\bar{d}(y,u)} - ae^{\bar{d}(x,u)} \leqslant (b - a)e^{\bar{d}(x,u)\vee\bar{d}(y,u)} \ ,$$

which shows (2.13). □

Proposition 2.4. *Let C be a normal cone in a Banach space $(X, \|\cdot\|)$. Consider $u \in C \setminus \{0\}$, and $\psi \in C^*$ with $\psi(u) > 0$. Then $(X_u, \|\cdot\|_u)$ is a Banach space. Moreover, $\|\cdot\|_u$ and $\bar{d}$ induce the same topology on C_u, and*

$$\|x - y\|_u \sim \bar{d}(x, y), \text{ when } x, y \to u \text{ in } (C_u, \bar{d}) \ , \tag{2.23}$$

$$\omega_u(x - y) \sim d(x, y), \text{ when } x, y \to u \text{ in } (\Sigma_u, \bar{d}) \ . \tag{2.24}$$

Also $C \cap X_u$ is a normal cone in $(X_u, \|\cdot\|_u)$ and has nonempty interior given by C_u. If C has nonempty interior in $(X, \|\cdot\|)$ and $u \in \operatorname{int} C$, then $X_u = X$ and $\|\cdot\|$ and $\|\cdot\|_u$ are equivalent norms on X.

Proof. It is proved in [Nus94, Proposition 1.1] that $\|\cdot\|_u$ and $\bar{d}$ induce the same topology on C_u. The fact that X_u is a complete metric space when C is normal is a result of Zabreĭko, Krasnosel′skiĭ, and Pokornyĭ [ZKP71] (see also Theorem 1.2 and Remark 1.1 in [Nus88]). This can also be derived from the completeness of $(C_u, \bar{d})$ (Proposition 2.1 above) and of the equivalence (2.23), which follows readily from (2.10) and (2.11). The equivalence (2.24) follows readily from (2.12) and (2.13). The fact that $C \cap X_u$ is a normal cone in $(X_u, \|\cdot\|_u)$ and has C_u as interior is remarked in [Nus94, Remark 1.1]. Indeed, since C is a proper cone and X_u is a vector space, $C \cap X_u$ is a proper cone. Moreover, from (2.6), $\|\cdot\|_u$ is order preserving on $C \cap X_u$, hence $C \cap X_u$ is normal. Also, by definition of C_u and (2.6), we get that C_u is open (if $au \leqslant x \leqslant bu$ with $a, b > 0$, then $B(x, \frac{a}{2}) \subset C_u$) and equal to the interior of $C \cap X_u$ (if $B(x, \varepsilon) \subset C \cap X_u$, then $\varepsilon u \leqslant x \leqslant \|x\|_u u$, and $x \in C_u$). Finally, it is proved in [Nus94, Proposition 1.1] that $\|\cdot\|_u$ and $\|\cdot\|$ are equivalent norms on $X = X_u$ when u is in the interior of C. □

We say that a cone C of a Banach space $(X, \|\cdot\|)$ is *reproducing* if $X = C - C := \{x - y \mid x, y \in C\}$. It is known that a cone C in a Banach space $(X, \|\cdot\|)$ with nonempty interior is reproducing (see for instance [AGN11, Proposition 2.4]).

Remark 2.5. It is shown in [Nus94] that the Thompson's metric coincides with the Finsler metric arising when considering the interior of the cone C as a manifold and equipping the tangent space at point u with the local norm $\|\cdot\|_u$. The Hilbert's metric arises in a similar way, when equipping the tangent space with the seminorm ω_u.

2.3. **AM-spaces with units.** Recall that an ordered set $(X, \leqslant)$ is a *sup-semilattice* (resp. *inf-semilattice*) if any nonempty finite subset F of X admits a least upper bound (resp. greatest lower bound) in X, denoted $\vee F$ (resp. $\wedge F$). We shall also use the infix notation $x \vee y = \vee\{x, y\}$ and $x \wedge y = \wedge\{x, y\}$. We say that X is a *lattice* when it is both a sup-semilattice and an inf-semilattice. A *Banach lattice* is a Banach space $(X, \|\cdot\|)$ equipped with an order relation, $\leqslant$, such that $(X, \leqslant)$ is a lattice, $x \leqslant y \Rightarrow x + z \leqslant y + z$ and $\lambda x \leqslant \lambda y$, for all $x, y, z \in X$ and $\lambda \geqslant 0$, and $|x| \leqslant |y| \Rightarrow \|x\| \leqslant \|y\|$, for all $x, y \in X$, where $|x| := x \vee (-x)$. Note that if X is a Banach lattice, then the lattice operations are continuous in the norm topology [MN91, Prop. 1.1.6]. Moreover, $X^+ := \{x \in X \mid 0 \leqslant x\}$ is closed and it is a reproducing normal cone. An *AM-space with unit* is a Banach lattice $(X, \|\cdot\|)$ such that $\|x \vee y\| = \|x\| \vee \|y\|$ for all $x, y \in X^+$, and such that the closed unit ball of X has a maximal element e, which is called the *unit* (see for instance [Sch74] or [AB99] for definitions and results about AM-spaces). Equivalently, X is an AM-space with unit, if X is a Banach lattice equipped with a distinguished element $e \geqslant 0$ (the unit), which is such that $\|x\| = \inf\{a \geqslant 0 \mid -ae \leqslant x \leqslant ae\}$ (see [AB99, Section 8.4], note however that in this reference, a slightly different definition of a unit is used, but it is shown that there exists another norm equivalent to $\|\cdot\|$ satisfying all the above conditions). The fundamental example of AM-space with unit is given by the space $\mathscr{C}(K)$ of continuous functions on a compact set K, equipped with the sup-norm, $\|\cdot\|_\infty$, the pointwise order $\leqslant$ and the unit $\mathbf{1}$, where $\mathbf{1}$ is the constant function $K \to \mathbb{R}$, $t \mapsto 1$. In fact, the Kakutani-Krein theorem ([AB99, Theorem 8.29] or [Sch74, Chapter II, Theorem 7.4]) shows that an AM-space with unit is isomorphic as an AM-space with unit (that is lattice isomorphic and isometric) to $\mathscr{C}(K)$, for some compact, Hausdorff set K.

Any AM-space with unit X can be put in isometric correspondence with the interior of a normal cone equipped with Thompson's metric, thanks to the following construction. Let $\imath : X \to \mathscr{C}(K)$ be the isomorphism (of AM-spaces with unit, where K is a compact set) given by Kakutani-Krein theorem. Consider $C = \mathscr{C}^+(K)$ the set of nonnegative continuous functions on K. Then, C is a normal cone, and the order $\leqslant_C$ associated to C is nothing but the pointwise order in $\mathscr{C}(K)$. The interior $\operatorname{int} C$ of C is the set of continuous functions $K \to \mathbb{R}$ that are bounded from below by a positive constant, or equivalently (since K is compact), the set of continuous functions $K \to \mathbb{R}$ that are positive everywhere. Consider the map $\log : \operatorname{int} C \to \mathscr{C}(K)$, which sends the continuous positive map $t \in K \mapsto y(t)$ to the continuous map $t \in K \mapsto \log y(t)$, and denote by $\exp = \log^{-1}$ its inverse. Then, the map $\imath^{-1} \circ \log : \operatorname{int} C \to X$ is an isometry between $\operatorname{int} C$ endowed with the Thompson's metric and X endowed with the distance associated with the norm $\|\cdot\|$.

3. Semidifferentiable maps

In this section, we consider (with a slight modification) the notion of semidifferentiability introduced by Penot in [Pen82] in the case of maps defined on Banach

spaces. We also refer the reader to [RW98] for the case of maps defined on $\mathbb{R}^n$. We establish here some properties of semidifferentiable maps defined on normed vector spaces, which will be needed in the proof of the main results and in the applications.

3.1. **Semidifferentiable maps on normed vector spaces.** Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed vector spaces, G be a subset of X, and $v \in G$. Recall that a map h from a cone C of X to Y is *homogeneous* if $f(ty) = tf(y)$, for all $t > 0$ and $y \in C$. We say that a map $f : G \to Y$ is *semidifferentiable* at v with respect to a cone $C \subset X$ if G contains a neighborhood of v in $v + C := \{v + x \mid x \in C\}$ (that is, if there exists $\varepsilon > 0$ such that all the elements of the form $v + x$ with $x \in C$ and $\|x\| \leqslant \varepsilon$ belong to G), and if there exists a continuous (positively) homogeneous map $h : C \to Y$ such that

$$f(v + x) = f(v) + h(x) + o(\|x\|), \text{ when } x \to 0, \ x \in C \ . \tag{3.1}$$

If $C = X$ (and v is in the interior of G), we say shortly that f is semidifferentiable at v. The map h, if it exists, is unique, since $h(x)$ must coincide with the classical (one sided) *directional derivative*:

$$f'_v(x) := \lim_{t\to 0^+} \frac{f(v + tx) - f(v)}{t}. \tag{3.2}$$

In fact, the definition (3.2) of directional derivatives is obtained by specializing (3.1) to a cone of the form $C = \{tx \mid t \geqslant 0\}$ and the classical definition of Frechet derivatives is obtained by specializing (3.1) to $C = X$, and requiring h to be linear. We call h the *semiderivative* of f at v with respect to C, and we denote it by f'_v.

The following characterization of semidifferentiable maps, which is a mere rephrasing of property (3.1), illuminates the requirements that semidifferentiability adds to the existence of directional derivatives.

Lemma 3.1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed vector spaces, G be a subset of X, $v \in G$, and C be a cone of X such that G contains a neighborhood of v in $v + C$. Then, a map $f : G \to Y$ is semidifferentiable at v with respect to C if, and only if,*

$$\frac{f(v + tx) - f(v)}{t} \to f'_v(x) \text{ when } t \to 0^+, \tag{3.3a}$$

$$\text{uniformly for } x \text{ in bounded sets of } C, \tag{3.3b}$$

$$\text{and } f'_v : C \to Y \text{ is continuous.}$$

The following lemma shows in particular that when f is locally Lipschitz continuous and X is finite dimensional, the last two properties are implied by the existence of directional derivatives.

Lemma 3.2. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed vector spaces, G be a subset of X, $v \in G$, and C be a cone of X such that G contains a neighborhood of v in $v+C$. Assume that $f : G \to Y$ is Lipschitz continuous in a neighborhood of v, and that f has directional derivatives at v with respect to all $x \in C$, so $f'_v : C \to Y$. Then f'_v is Lipschitz continuous. Moreover,*

$$\frac{f(v + tx) - f(v)}{t} \to f'_v(x) \text{ when } t \to 0^+, \tag{3.4a}$$

$$\text{uniformly for } x \text{ in compact sets of } C, \tag{3.4b}$$

and

$$\lim_{\substack{t\to 0^+ \\ x'\to x,\ x'\in C}} \frac{f(v+tx')-f(v)}{t} = f'_v(x) \quad \forall x\in C \ . \tag{3.5}$$

In particular, if X is finite dimensional and C is closed in X, then f is semidifferentiable at v with respect to C.

Proof. Since G contains a neighborhood of v in $v+C$ and f is Lipschitz continuous in a neighborhood of v, we can find $\varepsilon>0$ and $M\geqslant 0$ such that

$$(z,z'\in C \text{ and } \|z\|,\|z'\|\leqslant\epsilon) \implies \|f(v+z)-f(v+z')\|\leqslant M\|z-z'\| \ .$$

Let K denote a compact subset of C, let $R=\max\{\|x\| \mid x\in K\}$, and, for $0<t\leqslant R^{-1}\varepsilon$, consider the map $g_{v,t}: K\to Y$,

$$g_{v,t}(x)=\frac{f(v+tx)-f(v)}{t} \ .$$

For all $x,x'\in K$ and $0<t\leqslant R^{-1}\varepsilon$, we have

$$\|g_{v,t}(x)-g_{v,t}(x')\|\leqslant M\|x-x'\|. \tag{3.6}$$

The family $\{g_{v,t}\}_{0<t\leqslant R^{-1}\varepsilon}$ is an equicontinuous family of maps converging pointwise to f'_v on the compact set K, hence for all $x\in K$, the set $\{g_{v,t}(x) \mid 0<t\leqslant R^{-1}\varepsilon\}$ is relatively compact. This implies, by Ascoli's theorem, that $g_{v,t}$ converges uniformly to f'_v when $t\to 0^+$, which shows (3.4). Of course, by (3.6), f'_v is M-Lipschitz on K, and since this holds for all compact subsets $K\subset C$, f'_v is M-Lipschitz on C. Finally, to prove (3.5), it is enough to show that

$$\lim_{k\to\infty}\frac{f(v+t_kx_k)-f(v)}{t_k}=f'_v(x)$$

holds for all sequences $\{x_k\}_{k\geqslant 1}$, $x_k\in C$, $x_k\to x$, and $\{t_k\}_{k\geqslant 1}$, $t_k>0$, $t_k\to 0$. Since $\{x_k \mid k\geqslant 1\}\cup\{x\}$ is compact, it follows from (3.4) that

$$\lim_{k\to\infty}\left(\frac{f(v+t_kx_k)-f(v)}{t_k}-f'_v(x_k)\right)=0.$$

Since f'_v is continuous, $f'_v(x_k)\to f'_v(x)$, and we get (3.5). □

Remark 3.3. When $G=X=C=\mathbb{R}^n$ and $Y=\mathbb{R}$, Rockafellar and Wets [RW98, Chap. 7, §D] define semidifferentiable maps $f:\mathbb{R}^n\to\mathbb{R}$ at v by requiring that (3.5) holds, for all $x\in\mathbb{R}^n$. Assume now that X and Y are arbitrary normed vector spaces, that $C\subset X$ is a cone, that $G\subset X$, and $f: G\to X$. Then (3.5) holds if, and only if, f'_v is continuous and satisfies (3.4). (Indeed, we showed the "if" part in the proof of Lemma 3.2, and the "only if" part of the result is not difficult.) This equivalence is a special case of a general property, saying that a sequence of functions f_n (defined on a metric space X) converges *continuously* to a function f, i.e. satisfies $f_n(x_n)\to f(x)$, for all convergent sequences $x_n\to x$, if, and only if, f_n converges to f uniformly on compact sets and f is continuous, see [RW98, Theorem 7.14]. (The theorem of [RW98] is stated when $X=\mathbb{R}^n$ and $Y=\mathbb{R}$, but the property holds for arbitrary metric spaces X and Y.) When $X=C=\mathbb{R}^n$ (and $Y=\mathbb{R}$), Rockafellar and Wets show that (3.5) is equivalent to (3.1) [RW98, Theorem 7.21], so that our definition of semidifferentiable maps is consistent with the one of [RW98]. By comparing (3.3), which is equivalent to (3.1), with (3.4) together with the requirement that f'_v is continuous, which is equivalent to (3.5), we see that for general normed vector spaces, our definition (3.1) of semidifferentiable

maps becomes stronger than the one we would obtain by taking the definition (3.5) of [RW98].

We shall need the notion of *norm* of a continuous homogeneous map h from a cone C of a normed vector space $(X, \|\cdot\|)$ to a normed vector space $(Y, \|\cdot\|)$:

$$\|h\|_C := \sup_{x \in C \setminus \{0\}} \frac{\|h(x)\|}{\|x\|} < +\infty \ . \tag{3.7}$$

When C is obvious, and in particular when $C = X$, we will simply write $\|h\|$ instead of $\|h\|_C$. Since $h(0) = 0$ and h is continuous, there exists $\delta > 0$ such that $\|h(x)\| \leqslant 1$ for all $x \in C$ such that $\|x\| \leqslant \delta$. Hence, by homogeneity of h, $\|h\|_C \leqslant 1/\delta < +\infty$, as claimed in (3.7). In particular, when f is semidifferentiable at v with respect to C, there exists $\gamma \geqslant 0$ such that

$$\|f'_v(x)\| \leqslant \gamma \|x\| \quad \forall x \in C \ , \tag{3.8}$$

since by definition, f'_v is homogeneous and continuous. When f'_v is a linear map on X, that is, when f is differentiable, (3.8) implies that f'_v is Lipschitz continuous. Another condition which guarantees the Lipschitz continuity of f'_v was given in Lemma 3.2.

We next prove a chain rule for semidifferentiable maps:

Lemma 3.4 (Chain rule)**.** *Let $(X, \|\cdot\|)$, $(Y, \|\cdot\|)$ and $(Z, \|\cdot\|)$ be normed vector spaces, let G_1 and G_2 be subsets of X and Y respectively, let $f : G_1 \to Y$, $g : G_2 \to Z$ be two maps such that $f(G_1) \subset G_2$, and let $v \in G_1$. Assume that*

- (A1) *f is semidifferentiable at v with respect to a cone C_1;*
- (A2) *g is semidifferentiable at $f(v)$ with respect to a cone C_2;*
- (A3) *$f'_v(C_1) \subset C_2$ and $f(G_1) \subset f(v) + C_2$;*
- (A4) *$g'_{f(v)}$ is uniformly continuous on bounded sets of C_2.*

Then, $g \circ f : G_1 \to Z$ is semidifferentiable at v with respect to C_1, and

$$(g \circ f)'_v = g'_{f(v)} \circ f'_v \ .$$

Proof. Using Assumptions (A1) and (A2), we can write:

$$f(v + x) = f(v) + f'_v(x) + \|x\|\epsilon_1(x) \tag{3.9}$$

$$g(f(v) + y) = g(f(v)) + g'_{f(v)}(y) + \|y\|\epsilon_2(y) \tag{3.10}$$

where ϵ_1 (resp. ϵ_2) is a map defined on a neighborhood of 0 in C_1 (resp. C_2), with $\epsilon_1(x) \to 0$ when $\|x\| \to 0$ (resp. $\epsilon_2(y) \to 0$ when $\|y\| \to 0$). Using (3.9), (3.10) and Assumption (A3), we get,

$$g \circ f(v + x) = g \circ f(v) + g'_{f(v)}\big[f'_v(x) + \|x\|\epsilon_1(x)\big] + \eta(x) \ , \tag{3.11}$$

where $\eta(x) = \|f'_v(x) + \|x\|\epsilon_1(x)\|\epsilon_2\big[f'_v(x) + \|x\|\epsilon_1(x)\big]$. Using (3.8), we can write

$$\eta(x) = \|x\|\epsilon_3(x) \ , \tag{3.12}$$

where ϵ_3 is a map defined on a neighborhood of 0 in C_1, such that $\epsilon_3(x) \to 0$ when $\|x\| \to 0$. Using the homogeneity of $g'_{f(v)}$ and f'_v, we get:

$$g'_{f(v)}\big[f'_v(x) + \|x\|\epsilon_1(x)\big] = \|x\| g'_{f(v)}\big[f'_v(\|x\|^{-1}x) + \epsilon_1(x)\big] \ .$$

Hence, the uniform continuity assumption for $g'_{f(v)}$ (Assumption (A4)) implies that there is a map ϵ_4 defined on a neighborhood of 0 in C_1 such that

$$\begin{aligned} g'_{f(v)}\big[f'_v(x)+\|x\|\epsilon_1(x)\big] &= \|x\|\big[g'_{f(v)}\circ f'_v(\|x\|^{-1}x)+\epsilon_4(x)\big] \\ &= g'_{f(v)}\circ f'_v(x)+\|x\|\epsilon_4(x) \enspace , \end{aligned} \tag{3.13}$$

with $\epsilon_4(x)\to 0$ when $\|x\|\to 0$. Gathering (3.11),(3.12), and (3.13), we get

$$g\circ f(v+x)=g'_{f(v)}\circ f'_v(x)+\|x\|\epsilon_5(x) \enspace ,$$

where $\epsilon_5=\epsilon_3+\epsilon_4$, which concludes the proof of the lemma. □

Remark 3.5. When C_2 is closed, Assumption (A3) reduces to the condition $f(G_1)\subset f(v)+C_2$. Also, in this condition, G_1 can be replaced by a neighborhood of v in G_1.

Remark 3.6. By homogeneity of $g'_{f(v)}$, Assumption (A4) of Lemma 3.4 is equivalent to the uniform continuity of $g'_{f(v)}$ on the intersection of C_2 with the unit ball of Y.

Remark 3.7. Since a continuous map on a compact subset of a metric space is uniformly continuous, Assumption (A4) of Lemma 3.4 automatically holds when C_2 is closed and Y is finite dimensional. Therefore, Lemma 3.4 extends the result stated in [RW98, Exercise 10.27,(b)] in the case of finite dimensional vector spaces.

3.2. **Semidifferentiability of sups.** In applications to control and game theory, one needs to consider maps from $\mathbb{R}^n$ to $\mathbb{R}$ of the form

$$f=\sup_{a\in A} f_a \enspace , \tag{3.14}$$

where $(f_a)_{a\in A}$ is a family of maps $\mathbb{R}^n\to\mathbb{R}$, and the supremum is taken for the pointwise ordering of functions. Dually, f may be defined as the infimum of a family of maps.

We next give conditions which guarantee that a map of the form (3.14) has directional derivatives or that it is semidifferentiable. Recall that a real valued map $g:A\to\mathbb{R}$ is *upper-semicontinuous* (resp. *sup-compact*) if for all $\lambda\in\mathbb{R}$, the *upper level set* $S_\lambda(g)=\{a\in A\mid g(a)\geqslant\lambda\}$ is closed (resp. compact). The following theorem gives a general version, for semidifferentiable maps, of the rule of "differentiation" of a supremum, which has appeared in the literature in various guises, see Remarks 3.12, 3.14 below.

Theorem 3.8. *Let X denote a normed vector space, let $v,x\in X$, let V denote a neighborhood of v, and let $f:V\to\mathbb{R}$ be given by (3.14), where $(f_a)_{a\in A}$ is a family of maps $V\to\mathbb{R}$ and A is a Hausdorff topological space. Make the following assumptions:*

- (A1) *There exists $b\in A$ such that $f(v)=f_b(v)$;*
- (A2) *For all $a\in A$, the directional derivative of f at v in the direction x, $(f_a)'_v(x)$, exists;*
- (A3) *The maps $a\mapsto f_a(v)$ and $a\mapsto (f_a)'_v(x)$ are upper-semicontinuous;*
- (A4) *There exists a real number $t_0>0$ such that the map*

$$a\mapsto f_a(v)+t_0(f_a)'_v(x)$$

is sup-compact.

(A5) *We have*
$$f_a(v+tx) \leqslant f_a(v) + t(f_a)'_v(x) + t\epsilon_x(t) \quad \forall a \in A\ \forall t \in [0,t_1],$$
for some $t_1 > 0$ and some function $\epsilon_x : [0,t_1] \to \mathbb{R}$ independent of a, such that $\epsilon_x(t) \to 0$ when $t \to 0^+$;

Then, the directional derivative of f at v in the direction x exists, and:
$$f'_v(x) = \max_{a\in A,\ f_a(v)=f(v)} (f_a)'_v(x) \ . \tag{3.15}$$
Moreover, if the previous assumptions are satisfied for all $x \in X$, if X is finite dimensional and if f is Lipschitz continuous in a neighborhood of v, then, f is semidifferentiable at point v.

We shall see in Remarks 3.13–3.14 below that Assumptions (A2),(A5), or (A4) are implied by several standard assumptions, and that Theorem 3.8 extends several known results.

Before proving Theorem 3.8, we make the following observation.

Lemma 3.9. *Let* (A4) *and* (A3) *be as in Theorem 3.8. If Assumption* (A4) *is satisfied for some $t_0 > 0$, and if Assumption* (A3) *is satisfied, then Assumption* (A4) *is still satisfied if we replace t_0 by any $t \in (0,t_0)$.*

Proof. We first remark that if g, h are two maps $A \to \mathbb{R}$, such that g is sup-compact, h is upper-semicontinuous, and $h \leqslant g$, then h is also sup-compact. Indeed, for all $\lambda \in \mathbb{R}$, $S_\lambda(h)$ is included in $S_\lambda(g)$, $S_\lambda(h)$ is closed since h is upper-semicontinuous, and $S_\lambda(g)$ is compact because g is sup-compact, and thus, $S_\lambda(h)$ is compact.

Consider now $h_0 : a \mapsto f_a(v) + t_0(f_a)'(x)$. By Assumption (A4), h_0 is sup-compact. Take $t \in (0,t_0)$, and define $h_1 : a \mapsto f_a(v) + t(f_a)'(x)$, which is upper-semicontinuous thanks to Assumption (A3). We have $h_1(a) = \frac{t}{t_0}h_0(a) + (1 - \frac{t}{t_0})f_a(v) \leqslant \frac{t}{t_0}h_0(a) + (1-\frac{t}{t_0})f(v)$ which shows that h_1 is bounded from above by a sup-compact map. Thus, h_1 is sup-compact. □

Proof of Theorem 3.8. The first part of the theorem is a property of the one real variable functions $g(t) = f(v+tx)$ and $g_a(t) = f_a(v+tx)$, that is one can assume without loss of generality that $X = \mathbb{R}$, $v = 0$ and $x = 1$, $g = f$ and $g_a = f_a$, and omit x when possible.

By Assumption (A5), we have for all $t \in (0,t_1]$ and for all $a \in A$,
$$\frac{g_a(t)-g(0)}{t} = \frac{g_a(t)-g_a(0)+g_a(0)-g(0)}{t} \leqslant h_a(t) + \epsilon(t) \tag{3.16}$$
where
$$h_a(t) := \frac{t(g_a)'_0(1) + g_a(0) - g(0)}{t} \ .$$
Taking the sup of the inequalities (3.16), we get
$$\frac{g(t)-g(0)}{t} \leqslant \sup_{a\in A} h_a(t) + \epsilon(t) \ .$$
Since $g_a(0) \leqslant g(0)$, $h_a(t)$ is a nondecreasing function of t, so that
$$\limsup_{t\to 0^+} \frac{g(t)-g(0)}{t} \leqslant \lim_{t\to 0^+} \sup_{a\in A} h_a(t) + \epsilon(t) = \inf_{t>0} \sup_{a\in A} h_a(t) \ . \tag{3.17}$$
For all $t \in (0,t_0)$, the map $a \mapsto h_a(t)$, which is sup-compact by Lemma 3.9, attains its sup at some point $a_t \in A$. Let λ denote the value of the right hand side of (3.17).

We have, $h_{a_t}(t) \geqslant \lambda$, so that the upper level set $S_\lambda(a \mapsto h_a(t))$ is non-empty. Since a nonincreasing intersection of non-empty compact sets is non-empty, we can find $b \in \cap_{t>0} S_\lambda(a \mapsto h_a(t))$. Then, for all $t \in (0, t_0)$,

$$\frac{t(g_b)'_0(1) + g_b(0) - g(0)}{t} = (g_b)'_0(1) + \frac{g_b(0) - g(0)}{t} \geqslant \lambda \ . \tag{3.18}$$

Observe that λ is finite, because $\lambda \geqslant (g_b)'_0(1)$ with $g_b(0) = g(0)$ as in Assumption (A1). Thus, multiplying (3.18) by t and letting $t \to 0^+$, we get $g_b(0) - g(0) \geqslant 0$, and since the other inequality is obvious, $g_b(0) = g(0)$. Combining (3.18) and (3.17), we get

$$\limsup_{t\to 0^+} \frac{g(t) - g(0)}{t} \leqslant \inf_{t>0} \sup_{a\in A} h_a(t) = \lambda \leqslant (g_b)'_0(1) \leqslant \sup_{a\colon g_a(0)=g(0)} (g_a)'_0(1) \ . \tag{3.19}$$

Conversely, for all a such that $g_a(0) = g(0)$, we have $g(t) - g(0) = g(t) - g_a(0) \geqslant g_a(t) - g_a(0)$, since $g \geqslant g_a$, so that

$$\liminf_{t\to 0^+} \frac{g(t) - g(0)}{t} \geqslant \liminf_{t\to 0^+} \frac{g_a(t) - g_a(0)}{t} = (g_a)'_0(1) \ .$$

Thus,

$$\liminf_{t\to 0^+} \frac{g(t) - g(0)}{t} \geqslant \sup_{a\in A,\ g_a(0)=g(0)} (g_a)'_0(1) \ . \tag{3.20}$$

Gathering (3.19) and (3.20), we get that the directional derivative of f at v in the direction x, $f'_v(x)$, exists, is finite, and is given by (3.15). Moreover, the sup in (3.15) is a max since it is attained by taking $a = b$. □

Remark 3.10. Assumption (A4) is implied by standard assumptions.

First, if $a \to (f_a)'_v(x)$ is sup-compact, then Assumption (A4) follows from Assumption (A3). Indeed, for all $t_0 > 0$, $f_a(v) + t_0(f_a)'_v(x) \leqslant f(v) + t_0(f_a)'_v(x)$. Hence, $a \to f_a(v) + t_0(f_a)'_v(x)$ is sup-compact since it is upper-semicontinuous and bounded from above by a sup-compact map.

Symmetrically, if $a \mapsto f_a(v)$ is sup-compact and $a \to (f_a)'_v(x)$ is upper-semicontinuous and bounded from above, then again Assumption (A4) is satisfied for all $t_0 > 0$, since $f_a(v) + t_0(f_a)'_v(x) \leqslant f_a(v) + t_0\lambda$ for some $\lambda \in \mathbb{R}$.

Other assumptions which imply Assumption (A4) are the following. Assume that $a \mapsto f_a(v)$ is sup-compact, that Assumption (A3) is satisfied, and that $a \mapsto f_a(v) + t_1(f_a)'_v(x)$ is bounded from above, for some $t_1 > 0$. Then, we claim that $a \mapsto f_a(v) + t_0(f_a)'_v(x)$ is sup-compact, for all $t_0 \in (0, t_1)$. Indeed, let $\lambda = \sup_{a\in A} f_a(v) + t_1(f_a)'_v(x)$. Then, $f_a(v) + t_0(f_a)'_v(x) = (1 - \frac{t_0}{t_1})f_a(v) + \frac{t_0}{t_1}(f_a(v) + t_1(f_a)'_v(x)) \leqslant (1 - \frac{t_0}{t_1})f_a(v) + \frac{t_0}{t_1}\lambda$, and since a upper-semicontinuous map bounded from above by a sup-compact map is sup-compact, the claim is proved.

Remark 3.11. If $X = \mathbb{R}^n$, and if for all $a \in A$, $f_a : x \mapsto p_a \cdot x + r_a$ for some $p_a \in \mathbb{R}^n$ and $r_a \in \mathbb{R}$, then f_a is affine, thus differentiable at any $v \in \mathbb{R}^n$ with $(f_a)'_v(x) = p_a \cdot x$ for all $x \in \mathbb{R}^n$. Hence, f is convex, Assumption (A2) is satisfied, Assumption (A5) is satisfied with $\epsilon \equiv 0$ and Formula (3.15) becomes

$$f'_v(x) = \max_{a\in A,\ f_a(v)=f(v)} p_a \cdot x \ .$$

Assumption (A3) is satisfied when $a \mapsto p_a$ is continuous and $a \mapsto r_a$ is upper-semicontinuous. Finally, Assumption (A4) is satisfied if, and only if, $a \mapsto f_a(v+t_0x)$

is sup-compact. These assumptions are thus satisfied for instance when $a \mapsto r_a$ is sup-compact and $a \mapsto p_a$ is bounded and continuous.

Remark 3.12. From Remark 3.11, one can see that Theorem 3.8 extends the classical rule for the directional derivative of a convex function. When $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex, we can write, by Legendre-Fenchel duality [Roc70, Th. 12.2]:

$$f(x) = \sup_{p \in \operatorname{dom} f^*} p \cdot x - f^*(p) \ ,$$

where $f^*(p) = \sup_{x \in \mathbb{R}^n} p \cdot x - f(x)$ denotes the Legendre-Fenchel transform of f, evaluated at $p \in \mathbb{R}^n$, and $\operatorname{dom} f^* = \{p \in \mathbb{R}^n \mid f^*(p) < \infty\}$, so that the restriction of f to its domain can be written as (3.14) with $A = \operatorname{dom} f^*$, $a = p$, and $f_a(x) = f_p(x) = p \cdot x - f^*(p)$. Let us denote by V the interior of the domain of f, assume that V is nonempty and that $v \in V$. Assumption (A1) is satisfied, because $f(v) = p \cdot v - f^*(p) = f_p(v)$ holds for all p in the subdifferential $\partial f(v)$ of f at v, and $\partial f(v)$ is non-empty because f is convex and v is in V (see again [Roc70, Th. 23.4]). We are in the conditions of Remark 3.11, with $p_a = a$ and $r_a = -f^*(a)$. Hence, Assumptions (A2) and (A5) are satisfied. Since $a \mapsto p_a$ is continuous and $a \mapsto r_a$ is upper-semicontinuous, Assumption (A3) is satisfied. Moreover, Assumption (A4) is satisfied since for any $w \in V$, $a \mapsto f_a(w)$ is sup-compact and since $v + t_0 x \in V$ when $v \in V$, $x \in \mathbb{R}^n$ and $t_0 > 0$ is small enough. Indeed, $f_p(w) = p \cdot w - f^*(p) = p \cdot w - \sup_{x \in \mathbb{R}^n} p \cdot x - f(x) \leqslant p \cdot (w - x) + f(x)$, for all $x \in \mathbb{R}^n$. Taking $\epsilon > 0$ and $x = w + \epsilon \frac{p}{\|p\|}$, we get $f_p(w) \leqslant -\epsilon p \cdot \frac{p}{\|p\|} + f(w + \epsilon \frac{p}{\|p\|}) = -\epsilon \|p\| + f(w + \epsilon \frac{p}{\|p\|})$. Since f is convex and $w \in V$, f is continuous on a neighborhood of w [Roc70] and thus for $\epsilon > 0$ small enough $f(w + \epsilon \frac{p}{\|p\|})$ can be bounded independently of p, which implies that the upper level sets of $p \mapsto f_p(w)$ are bounded. Since these level sets are closed subsets of $\mathbb{R}^n$, they are compact, so $p \mapsto f_p(w)$ is sup-compact. Hence all the assumptions of Theorem 3.8 are satisfied. Finally, Formula (3.15) becomes

$$f'_v(x) = \max_{p \in \partial f(v)} p \cdot x \ ,$$

which coincides with the classical formula of Theorem 23.4 of [Roc70].

Remark 3.13. When $X = \mathbb{R}^n$, and $f_a : V \to \mathbb{R}$ is concave, the directional derivative

$$(f_a)'_v(x) = \lim_{t \to 0^+} \frac{f_a(v + tx) - f_a(v)}{t} = \sup_{t > 0} \frac{f_a(v + tx) - f_a(v)}{t}$$

exists and is finite [Roc70, Th 23.1 and 23.4] (see also Remark 3.12), so that Assumption (A2) is satisfied. Then, Assumption (A5) is satisfied with $\epsilon \equiv 0$.

Remark 3.14. When A is finite, only Assumption (A2) has to be checked in the first part of Theorem 3.8 and the second part of Theorem 3.8 shows that semidifferentiable maps $\mathbb{R}^n \to \mathbb{R}$ are stable by max, a well known fact [RW98, Exercise 10.27]. More generally, the part of [RW98, Theorem 10.31] which asserts that lower-$\mathscr{C}_1$ maps are semidifferentiable may be recovered from Theorem 3.8 (in [RW98], a map $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *lower-*$\mathscr{C}_1$ if it can be written as (3.14) with A compact, $x \mapsto f_a(x)$ of class $\mathscr{C}_1$, and $(a, x) \mapsto (f_a(x), (f_a)'_x)$ continuous.)

Remark 3.15. The following counter example shows that the upper-semicontinuity assumptions are useful in Theorem 3.8. Take any function $f : \mathbb{R} \to \mathbb{R}$ of Lipschitz constant 1, which is not semidifferentiable (for instance, $f(x) = \frac{x}{2} \sin(\log |x|)$, which

is not semidifferentiable at 0). Then, $f :]-1,1[\to \mathbb{R}$ can be written as (3.14) with $A = [-1,1]$ and

$$f_a(x) = f(a) - |x-a| \ .$$

All the assumptions of Theorem 3.8 are satisfied, except the requirement that $a \mapsto (f_a)'_v(x)$ be upper semicontinuous. Indeed,

$$(f_a)'_v(x) = \begin{cases} -x & \text{if } v > a \\ x & \text{if } v < a \\ -|x| & \text{if } v = a, \end{cases}$$

so that $(f_v)'_v(x) = -|x| < \limsup_{a\to x}(f_a)'_v(x) = +|x|$, unless $x = 0$.

4. Semiderivatives of order preserving and nonexpansive maps

In this section, we consider various classes of maps on cones, and establish auxiliary results concerning their semidifferentials.

Let $(X, \|\cdot\|)$ be a Banach space endowed with a partial ordering $\leqslant$, and f be a map from a subset $D \subset X$, to X. Recall that f is *order-preserving* if for all $x, y \in D$, $x \leqslant y \implies f(x) \leqslant f(y)$. We shall say that f is *convex* if $f((1-t)x+ty) \leqslant (1-t)f(x) + tf(y)$ for all $0 \leqslant t \leqslant 1$ and $x, y \in D$, such that $(1-t)x + ty \in D$. We shall say that f is *subhomogeneous* if $tf(y) \leqslant f(ty)$ for all $0 \leqslant t \leqslant 1$ and $y \in D$, such that $ty \in D$. Also, we shall say that f satisfies a property (for instance is order preserving, or homogeneous,...) in a neighborhood of a point v of D, if there exists a neighborhood V of v in D such that $f|_V$ satisfies this property. We shall use systematically the following well known elementary properties (see for instance [Nus88, Tho63, Bus73, Pot77]):

Lemma 4.1. *Let C be a proper cone, $u \in C \setminus \{0\}$, $\psi \in C^* \setminus \{0\}$ such that $\psi(u) > 0$ and let $\Sigma_u = \{x \in C_u \mid \psi(x) = \psi(u)\}$.*

- (i) *If $f : C_u \to C$ is order-preserving and homogeneous, then $f(C_u) \subset C_{f(u)}$ and f is nonexpansive with respect to d and $\bar{d}$.*
- (ii) *If $f : C_u \to C$ is order preserving and subhomogeneous, then $f(C_u) \subset C_{f(u)}$ and f is nonexpansive with respect to $\bar{d}$, and the restriction $f|_{\Sigma_u}$ of f to Σ_u is nonexpansive with respect to d.*

We now introduce additive analogues of (sub-) homogeneity: we shall say that a self map h of a Banach space $(X, \|\cdot\|)$ is *additively homogeneous* with respect to $v \in X$ if, for all $x \in X$ and $t \in \mathbb{R}$, $h(x + tv) = h(x) + tv$; similarly, we shall say that h is *additively subhomogeneous* with respect to v if, for all $x \in X$ and $t \geqslant 0$, $h(x + tv) \leqslant h(x) + tv$. The following lemma relates the properties of f with those of f'.

Lemma 4.2. *Let C be a proper cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let G be an open subset of $(X, \|\cdot\|)$ included in C and $f : G \to \operatorname{int} C$. Let $v \in G$ be a fixed point of f: $f(v) = v$. Let $\psi \in C^* \setminus \{0\}$ be such that $\psi(v) = 1$ and denote $\Sigma = \{x \in \operatorname{int} C \mid \psi(x) = 1\}$. Assume that f is semidifferentiable at v. The following implications hold:*

- (i) *If f is order preserving in a neighborhood of v, then $f'_v : X \to X$ is order preserving.*
- (ii) *If f is convex in a neighborhood of v, then $f'_v : X \to X$ is convex.*

(iii) *If f is homogeneous in a neighborhood of v, then f'_v is additively homogeneous with respect to v.*
(iv) *If f is subhomogeneous in a neighborhood of v, then f'_v is additively subhomogeneous with respect to v.*
(v) *Assume that there exist $\delta > 0$ such that $f(tv) \leqslant tf(v)$ for all $1 \leqslant t \leqslant 1+\delta$. Then, $f'_v(v) \leqslant v$.*
(vi) *Assume that there exist $\delta > 0$ such that $\delta \leqslant 1$ and $tf(v) \leqslant f(tv)$ for all $1-\delta \leqslant t \leqslant 1$. Then, $f'_v(-v) \geqslant -v$.*
(vii) *If f is nonexpansive with respect to $\bar{d}$ in a neighborhood of v, then f'_v is nonexpansive with respect to $\|\cdot\|_v$.*
(viii) *If $f|_{G\cap\Sigma}$ is nonexpansive with respect to d in a neighborhood of v, then $f'_v|_{\psi^{-1}(0)}$ is nonexpansive with respect to ω_v.*

Proof. We shall only need the definition (3.2) of f'_v.

(i): If f is order preserving in a neighborhood of v, then for all $x, y \in X$ such that $x \leqslant y$, we have $f(v+tx) \leqslant f(v+ty)$ for all $t > 0$ small enough, hence, from (3.2), $f'_v(x) \leqslant f'_v(y)$, which shows that f'_v is order preserving.

(ii): If f is convex in a neighborhood of v, then for all $x, y \in X$ and $s \in [0,1]$, we have $f(v+t((1-s)x+sy)) = f((1-s)(v+tx)+s(v+ty)) \leqslant (1-s)f(v+tx)+sf(v+ty)$ for all $t > 0$ small enough, hence, from (3.2), $f'_v((1-s)x+sy) \leqslant (1-s)f'_v(x)+sf'_v(y)$, which shows that f'_v is convex.

(iii): If f is homogeneous in a neighborhood of v, then for all $x \in X$, $s \in \mathbb{R}$,

$$f(v+t(x+sv)) = f((1+ts)(v+\frac{t}{1+ts}x)) = (1+ts)f(v+\frac{t}{1+ts}x)$$

for all $t > 0$ small enough. Since $f(v) = v$, this leads to

$$\frac{f(v+t(x+sv))-f(v)}{t} = \frac{1+ts}{t}\left(f(v+\frac{t}{1+ts}x)-f(v)\right)+sv \tag{4.1}$$

for all $t > 0$ small enough. Using (3.2) and (4.1), we get

$$f'_v(x+sv) = f'_v(x)+sv$$

which shows that f'_v is additively homogeneous with respect to v.

(iv): If f is subhomogeneous in a neighborhood of v, then $f(tx) \leqslant tf(x)$ for all $x \in X$ and $t \geqslant 1$. By the same arguments as for (iii), we obtain that for all $x \in X$, $s \geqslant 0$,

$$f(v+t(x+sv)) \leqslant (1+ts)f(v+\frac{t}{1+ts}x)$$

and

$$\frac{f(v+t(x+sv))-f(v)}{t} \leqslant \frac{1+ts}{t}\left(f(v+\frac{t}{1+ts}x)-f(v)\right)+sv \tag{4.2}$$

for all $t > 0$ small enough. Using (3.2) and (4.2), we get

$$f'_v(x+sv) \leqslant f'_v(x)+sv$$

for all $s \geqslant 0$, which shows that f'_v is additively subhomogeneous with respect to v.

(v): Assume that there exist $\delta > 0$ such that $f(tv) \leqslant tf(v)$ for all $1 \leqslant t \leqslant 1+\delta$. Then, using $f(v) = v$, we get for all $0 < s \leqslant \delta$

$$\frac{f(v+sv)-f(v)}{s} \leqslant f(v) = v$$

and passing to the limit when s goes to 0, we obtain $f'_v(v) \leqslant v$.

(vi): Assume that there exist $\delta > 0$ such that $\delta \leqslant 1$ and $tf(v) \leqslant f(tv)$ for all $1-\delta \leqslant t \leqslant 1$. Then, similarly to case (v), we get for all $0 < s \leqslant \delta$

$$\frac{f(v-sv)-f(v)}{s} \geqslant -f(v) = -v$$

and passing to the limit when s goes to 0, we obtain $f'_v(-v) \geqslant -v$.

(vii): If f is nonexpansive with respect to $\bar{d}$ in a neighborhood of v, then for all $x, y \in X$ such that $x \neq y$,

$$\frac{\bar{d}(f(v+tx), f(v+ty))}{\bar{d}(v+tx, v+ty)} \leqslant 1 \tag{4.3}$$

for all $t > 0$ small enough. In particular $f(v+tx)$ and $f(v+ty)$ tend to v in $(\operatorname{int} C, \bar{d})$ when $t \to 0^+$. Using (2.23) and (4.3), we get

$$\lim_{t\to 0^+} \frac{\|f(v+tx)-f(v+ty)\|_v}{\|tx-ty\|_v} \leqslant 1 \enspace .$$

Using (3.2), we deduce

$$\frac{\|f'_v(x)-f'_v(y)\|_v}{\|x-y\|_v} \leqslant 1$$

which shows that f'_v is nonexpansive with respect to $\|\cdot\|_v$.

(viii): If $f|_{G\cap\Sigma}$ is nonexpansive with respect to d in a neighborhood of v, then, for all $x, y \in \psi^{-1}(0)$ such that $x \neq y$,

$$\frac{d(f(v+tx), f(v+ty))}{d(v+tx, v+ty)} \leqslant 1$$

for all $t > 0$ small enough (since $v+tx$ and $v+ty \in G \cap \Sigma$ for all $t > 0$ small enough). Hence, using (2.24), we get

$$\lim_{t\to 0^+} \frac{\omega_v(f(v+tx)-f(v+ty))}{\omega_v(tx-ty)} \leqslant 1 \enspace .$$

Using (3.2), we deduce

$$\frac{\omega_v(f'_v(x)-f'_v(y))}{\omega_v(x-y)} \leqslant 1$$

which shows that $f'_v|_{\psi^{-1}(0)}$ is nonexpansive with respect to ω_v. □

We shall also need:

Lemma 4.3. *Let h denote a self-map of a Banach space $(X, \|\cdot\|)$ endowed with a partial ordering $\leqslant$, and $v \in X$. If h is convex, homogeneous, and if $h(v) \leqslant v$, then h is additively subhomogeneous with respect to v.*

Proof. If h is convex and $h(v) \leqslant v$, then using the homogeneity of h, we get for all $t \geqslant 0$ and $x \in X$,

$$h(x+tv) \leqslant \frac{1}{2}(h(2x)+h(2tv)) = h(x)+th(v) \leqslant h(x)+tv \enspace ,$$

which shows that h is additively subhomogeneous with respect to v. □

Symmetrically, if $h : X \to X$ is concave, homogeneous and such that $h(-v) \geqslant -v$, then h is additively subhomogeneous with respect to v. Lemma 4.3 covers only a special case: a homogeneous and additively subhomogeneous map need not be convex or concave (consider for instance $h : \mathbb{R}^2 \to \mathbb{R}^2, h(x_1, x_2) = ((x_1 \vee x_2) \wedge$

$x_1/2, x_2)$, which is homogeneous and additively subhomogeneous with respect to $(1,1)$).

If C, G, f are as in Lemma 4.2, to study the eigenvectors of f, we shall pick a linear form $\psi \in C^* \setminus \{0\}$, define the set $\Sigma = \{x \in \operatorname{int} C \mid \psi(x) = 1\}$, and consider the map

$$\tilde{f} : G \to \Sigma, \quad \tilde{f}(x) = \frac{f(x)}{\psi(f(x))} \ . \tag{4.4}$$

The following lemma states some basic properties of $\tilde{f}$.

Lemma 4.4. *Let C be a proper cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let G be an open subset of $(X, \|\cdot\|)$ included in C and $f : G \to \operatorname{int} C$. Let $\psi \in C^* \setminus \{0\}$, $\Sigma = \{x \in \operatorname{int} C \mid \psi(x) = 1\}$, let $\tilde{f}$ be defined by (4.4) and $g = \tilde{f}|_{G\cap\Sigma}$. If $f|_{G\cap\Sigma}$ is nonexpansive with respect to d, so is g. Let $v \in G \cap \Sigma$ be a fixed point of f: $f(v) = v$. Assume that f is semidifferentiable at v. Then, $\tilde{f}$ is semidifferentiable at v, and*

$$\tilde{f}'_v(x) = f'_v(x) - \psi(f'_v(x))v \quad \forall x \in X \ . \tag{4.5}$$

Moreover, if $f|_{G\cap\Sigma}$ is nonexpansive with respect to d in a neighborhood of v, then $g'_v = \tilde{f}'_v|_{\psi^{-1}(0)}$ is nonexpansive with respect to ω_v.

Proof. By definition of d, we have $d(\lambda x, \mu y) = d(x, y)$ for all $\lambda, \mu > 0$ and $x, y \in \operatorname{int} C$. Hence, if $f|_{G\cap\Sigma}$ is nonexpansive with respect to d, then for all $x, y \in G \cap \Sigma$,

$$d(g(x), g(y)) = d\Big(\frac{f(x)}{\psi(f(x))}, \frac{f(y)}{\psi(f(y))}\Big) = d(f(x), f(y)) \leqslant d(x, y) \ ,$$

which shows that g is nonexpansive with respect to d.

Consider the map $R : \operatorname{int} C \to \Sigma$, $y \mapsto \frac{y}{\psi(y)}$. We have $\tilde{f} = R \circ f$. Since ψ is linear, thus differentiable at any point $w \in X$, and $\psi(y) > 0$ for all $y \in \operatorname{int} C$, it follows that R is differentiable at any $w \in \operatorname{int} C$, with

$$R'_w(y) = \frac{y}{\psi(w)} - \frac{\psi(y)w}{\psi(w)^2} \ . \tag{4.6}$$

Since $f(v) = v$, f is semidifferentiable at v, and R is differentiable at $f(v) = v$. In particular, R'_v is uniformly continuous on all bounded sets, and it follows from Lemma 3.4 that $\tilde{f}$ is semidifferentiable at v and that $\tilde{f}'_v = R'_v \circ f'_v$. From (4.6) and $\psi(v) = 1$, we get (4.5).

If $f|_{G\cap\Sigma}$ is nonexpansive with respect to d in a neighborhood of v, then by the same arguments as above, $g = \tilde{f}|_{G\cap\Sigma}$ is nonexpansive with respect to d in a neighborhood of v, and by Lemma 4.2, (viii) applied to $\tilde{f}$, $g'_v = \tilde{f}'_v|_{\psi^{-1}(0)}$ is nonexpansive with respect to ω_v. □

The following result, which controls $x - h(x)$ in terms of the fixed point set of an order preserving additively subhomogeneous map h, will play a key role in the proof of the uniqueness theorem for eigenvectors (Theorem 7.5 below),

Lemma 4.5. *Let C be a proper cone with nonempty interior in a Banach space $(X, \|\cdot\|)$ and let $v \in \operatorname{int} C$. Let $h : X \to X$ be order preserving and additively subhomogeneous with respect to v. Assume that the set $S = \{y \in X \mid h(y) = y\}$ of*

fixed points of h is nonempty. We have, for all $x \in X$

$$m(x - h(x) \,/\, v) \leqslant \inf_{y \in S} m(x - y \,/\, v) \vee 0 \tag{4.7}$$

$$M(x - h(x) \,/\, v) \geqslant \sup_{y \in S} M(x - y \,/\, v) \wedge 0 \ . \tag{4.8}$$

Proof. We first show that if $h(0) = 0$, then

$$M(x - h(x) \,/\, v) \geqslant M(x \,/\, v) \wedge 0 \quad \forall x \in X \ . \tag{4.9}$$

Let $x \in X$ and denote $\beta = M(x \,/\, v)$ and $z = x - h(x)$. Then, $x \leqslant \beta v \leqslant (\beta \vee 0)v$. Since $h(0) = 0$ and h is order preserving and additively subhomogeneous with respect to v, we get $h(x) \leqslant h((\beta \vee 0)v) \leqslant h(0) + (\beta \vee 0)v = (\beta \vee 0)v$. Hence,

$$x = z + h(x) \leqslant M(z \,/\, v)v + (\beta \vee 0)v = (M(z \,/\, v) + \beta \vee 0)v \ . \tag{4.10}$$

Since $\beta = \inf\{a \in \mathbb{R} \mid x \leqslant av\}$, we deduce from (4.10) that $\beta \leqslant M(z \,/\, v) + \beta \vee 0$, thus $\beta \wedge 0 = \beta - \beta \vee 0 \leqslant M(z \,/\, v)$, which shows (4.9).

Applying (4.9) to $-x$ and replacing h by $h^-(x) = -h(-x)$, which is order preserving, additively subhomogeneous with respect to v and satisfies $h^-(0) = 0$, we get $M(-x + h(x) \,/\, v) \geqslant M(-x \,/\, v) \wedge 0$. Since $m(x \,/\, v) = -M(-x \,/\, v)$, this shows that

$$m(x - h(x) \,/\, v) \leqslant m(x \,/\, v) \vee 0 \quad \forall x \in X \ . \tag{4.11}$$

Let $y \in S$ and consider $h_y : X \to X$, $x \mapsto h_y(x) = h(y + x) - y$. Then, $h_y(0) = 0$ and h_y is order preserving and additively subhomogeneous with respect to v: $h_y(x + tv) = h(y + x + tv) - y \leqslant h(y + x) + tv - y = h_y(x) + tv$ for all $t \geqslant 0$. Replacing h by h_y in (4.9), we get

$$M(x + y - h(x + y) \,/\, v) \geqslant M(x \,/\, v) \wedge 0 \quad \forall x \in X \ ,$$

then, replacing x by $x - y$, we obtain

$$M(x - h(x) \,/\, v) \geqslant M(x - y \,/\, v) \wedge 0 \quad \forall x \in X \ .$$

Taking the supremum with respect to $y \in S$ leads to (4.8). Similarly, applying (4.11) to h_y and replacing x by $x - y$, we obtain (4.7). □

Let $(X, \|\cdot\|)$ be an AM-space with unit, denoted by e, and let $f : X \to X$ be a map. We shall say that f is *additively homogeneous* (resp. *additively subhomogeneous*) if f is additively homogeneous (resp. subhomogeneous) with respect to e, that is (see above) $f(x + te) = f(x) + te$ (resp. $f(x + te) \leqslant f(x) + te$), for all $x \in X$ and $t \in \mathbb{R}$ (resp. $t \geqslant 0$). We call an *additive eigenvector* of f a vector $x \in X$ such that $f(x) = x + \lambda e$, for some $\lambda \in \mathbb{R}$. We denote by ω the seminorm ω_e defined in (2.7).

Additive versions of some results of Lemmas 4.1 and 4.2 are easy to check and are given without proof (a variant of Lemma 4.6 can be found in [CT80]).

Lemma 4.6 (Compare with [CT80])**.** *Let $(X, \|\cdot\|)$ be an AM-space with unit, denoted by e, and ω denotes the seminorm ω_e defined in* (2.7)*. Let $\psi \in (X^+)^* \setminus \{0\}$.*

(i) *If $F : X \to X$ is order-preserving and additively homogeneous, then F is nonexpansive with respect to $\|\cdot\|$ and ω.*
(ii) *If $F : X \to X$ is order preserving and additively subhomogeneous, then F is nonexpansive with respect to $\|\cdot\|$, and the restriction $F|_{\psi^{-1}(0)}$ of F to $\psi^{-1}(0)$ is nonexpansive with respect to ω.*

Lemma 4.7. *Let $(X, \|\cdot\|)$ be an AM-space with unit. Let G be an open subset of X and $F : G \to X$. Let $v \in G$ be a fixed point of F: $F(v) = v$. Assume that F is semidifferentiable at v. The following implications hold:*

(i) *If F is order preserving in a neighborhood of v, then $F'_v : X \to X$ is order preserving.*
(ii) *If F is convex in a neighborhood of v, then $F'_v : X \to X$ is convex.*
(iii) *If F is additively homogeneous in a neighborhood of v, then F'_v is additively homogeneous.*
(iv) *If F is additively subhomogeneous in a neighborhood of v, then F'_v is additively subhomogeneous.*
(v) *If F is nonexpansive with respect to $\|\cdot\|$ in a neighborhood of v, then F'_v is nonexpansive with respect to $\|\cdot\|$.*

5. Spectral radius notions and non-linear Fredholm property

Some of our main results rely on some mild compactness (nonlinear Fredholm type) condition. In order to discuss it, we first recall the definition of several notions of non-linear spectral radius, as well as some results of [MPN02, AGN11], on generalized measures of noncompactness.

5.1. **Spectral radius, measures of noncompactness, and essential spectral radius.** Let C be a cone of a Banach space $(X, \|\cdot\|)$ and h be a map, homogeneous (of degree 1) and continuous, from C to C. Following [MPN02], we define:

$$\tilde{r}_C(h) = \lim_{k\to\infty} \|h^k\|_C^{1/k} = \inf_{k\geqslant 1} \|h^k\|_C^{1/k} \qquad \text{where } \|h^k\|_C \text{ is defined in (3.7)}, \tag{5.1}$$

When C is obvious and in particular when $C = X$, C will be omitted in the previous notations. Since h is continuous at 0, we have $0 \leqslant \tilde{r}_C(h) < +\infty$. The equality of the limit and the infimum in (5.1) follows from $\|h^{k+\ell}\| \leqslant \|h^k\|\|h^\ell\|$. The number $\tilde{r}_C(h)$ is called the *Bonsall's cone spectral radius* of h. If now C has nonempty interior $\operatorname{int} C$, we define another spectral radius:

$$\mathrm{cw}_C(h) = \inf\{\lambda > 0 \mid \exists x \in \operatorname{int} C,\ h(x) \leqslant \lambda x\} \ . \tag{5.2}$$

In [MPN02] and then in [AGN11] these spectral radius are compared with other notions of spectral radius like the cone spectral radius and the cone eigenvalue spectral radius. We recall bellow some of the results of [AGN11] that are needed in the following sections.

A map ν from the set of bounded subsets of X to the set of real nonnegative numbers is called a *homogeneous generalized measure of noncompactness* if for all bounded subsets A, B of X and for all real scalars λ,

$$\nu(A) = 0 \Leftrightarrow \operatorname{clo} A \text{ is compact} \tag{5.3a}$$

$$\nu(A + B) \leqslant \nu(A) + \nu(B) \tag{5.3b}$$

$$\nu(\operatorname{clo}\operatorname{conv}(A)) = \nu(A) \tag{5.3c}$$

$$\nu(\lambda A) = |\lambda|\nu(A) \tag{5.3d}$$

$$A \subset B \implies \nu(A) \leqslant \nu(B) \ . \tag{5.3e}$$

We use the notation $\operatorname{clo} A$ for the closure of a set A and $\operatorname{conv} A$ for its convex hull. Note that equations (5.3a)–(5.3e) imply that $\nu(A \cup K) = \nu(A)$ whenever A is a bounded subset of X and K is a compact subset of X, see e.g. Prop. 3.3 in [AGN11].

For every bounded subset A of X, let $\alpha(A)$ denote the infimum of all $\delta > 0$ such that there exists an integer k and k subsets $S_1, \dots, S_k \subset A$ of diameter at most δ, such that $A = S_1 \cup \cdots \cup S_k$. The map α, introduced by Kuratowski and further studied by Darbo (see [MPN02] for references), is a particular case of a homogeneous generalized measure of noncompactness.

If $h : D \subset X \to X$ is a map sending bounded sets to bounded sets, and ν is a homogeneous generalized measure of noncompactness, we define

$$\nu_D(h) = \inf\{\lambda > 0 \mid \nu(h(A)) \leqslant \lambda\nu(A), \text{ for all bounded sets } A \subset D\} .$$

If in addition $h(D) \subset D$, we define :

$$\rho_D(h) = \lim_{k\to\infty} (\nu_D(h^k))^{1/k} = \inf_{k\geqslant 1} \nu_D(h^k)^{1/k} .$$

If C is a cone and h: $C \to C$ is homogeneous and Lipschitz continuous with constant κ, then $\alpha_C(h) \leqslant \kappa$. A general map h: $D \subset X \to X$, such that $\nu_D(h) \leqslant k < 1$ is called a *k-set contraction* (with respect to ν). If C is a cone and $h : C \to C$ is homogeneous, $\rho_C(h)$ is called the *cone essential spectral radius* of h associated to the homogeneous generalized measure of noncompactness ν. (We note that a more refined notion of cone essential spectral radius, which avoids various pathologies which can occur with the above definition, has been recently developed in [MPN10]. See also [MPN11].)

We saw in Lemma 4.2 that several elementary properties (including monotonicity) of a map carry over to its semidifferential. The same turns out to be true for properties involving k-set contractions.

Proposition 5.1. *Let $f : G \to X$ be a map, and let $v \in G$ be a fixed point of $f : f(v) = v$. Assume that f is semidifferentiable at v with respect to a closed cone C. Then, for any relative neighborhood U of v in $v + C$ such that $U \subset G$ and for any generalized measure of noncompactness ν on X, we have*

$$\nu_C(f'_v) \leqslant \nu_U(f) . \tag{5.4a}$$

Moreover, if there exists a neighborhood U of v in $v + C$, such that $U \subset G$ and $f(U) \subset U$, then f'_v sends C to itself, and if in addition f'_v is uniformly continuous on bounded sets of C, we have

$$\rho_C(f'_v) \leqslant \rho_U(f) . \tag{5.4b}$$

Proof. Let us prove (5.4a). Let S be a bounded subset of C. Since f is a semidifferentiable at v with respect to C, there exists a neighborhood U of v in $v + C$, such that $U \subset G$ and for all such a neighborhood U, there exists $t_0 > 0$ such that $v + tS \subset U$ for $0 \leqslant t \leqslant t_0$, since S is bounded in C. Moreover, from Lemma 3.1 it follows that

$$\varepsilon(t) := \sup_{x\in S} t^{-1}\|f(v + tx) - f(v) - tf'_v(x)\| \to 0 \text{ when } t \to 0_+ .$$

Then for all $t > 0$ such that $t \leqslant t_0$, we have

$$tf'_v(x) - f(v + tx) + f(v) \in B(0, t\varepsilon(t)), \text{ for all } x \in S ,$$

which implies that

$$tf'_v(S) \subset f(v + tS) - f(v) + B(0, t\varepsilon(t)) .$$

Applying ν, and using Properties (5.3d), (5.3e), (5.3b), (5.3a), we get

$$\begin{aligned} t\nu(f'_v(S)) &= \nu(tf'_v(S)) \\ &\leqslant \nu(f(v+tS) - f(v) + B(0, t\varepsilon(t))) \\ &\leqslant \nu(f(v+tS)) + \nu(\{-f(v)\}) + \nu(B(0, t\varepsilon(t))) \\ &\leqslant \nu(f(v+tS)) + t\varepsilon(t)\nu(B(0,1)) \ . \end{aligned} \tag{5.5}$$

Since $v + tS \subset U$ for $t \leqslant t_0$, we obtain, by definition of $\nu_U(f)$,

$$\nu(f(v+tS)) \leqslant \nu_U(f)\nu(v+tS) \ .$$

By (5.3b), (5.3a), and (5.3d), $\nu(v+tS) \leqslant \nu(\{v\}) + \nu(tS) = \nu(tS) = t\nu(S)$. We now get from (5.5):

$$\nu(f'_v(S)) \leqslant \nu_U(f)\nu(S) + \varepsilon(t)\nu(B(0,1))$$

for all t small enough, hence $\nu(f'_v(S)) \leqslant \nu_U(f)\nu(S)$, which shows (5.4a).

Assume now that there exists a neighborhood U of v in $v + C$, such that $U \subset G$ and $f(U) \subset U$. Then, $f(U) \subset U \subset v + C$, and since C is closed, we deduce that $f'_v(C) \subset C$. Assume in addition that f'_v is uniformly continuous on bounded sets of C and let us prove (5.4b). Since f'_v is uniformly continuous on bounded sets of C, so is $(f'_v)^n$, since, by (3.8), f'_v sends bounded sets to bounded sets. Using the chain rule (Lemma 3.4), we obtain by induction that f^n is semidifferentiable at v with respect to C, and that $(f^n)'_v = (f'_v)^n$. Then, applying (5.4a) to f^n, we get $\nu_C((f'_v)^n) \leqslant \nu_U(f^n)$. Taking the $1/n$ power and then the infimum over all $n \geqslant 1$, we get $\rho_C(f'_v) \leqslant \rho_U(f)$. □

5.2. **A nonlinear Fredholm-type property.** We now introduce a nonlinear Fredholm-type property, also considered in [AGN11], which will be required to establish the uniqueness result for fixed point and eigenvectors in Sections 6–7. If $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ are Banach spaces, D is a subset of X, and $g : D \to Y$ is a map, we shall say that g has *Property* (F) when

(F) any sequence $\langle x_j \in D \mid j \geqslant 1\rangle$, bounded in X, and such that $g(x_j) \to_{j\to\infty} 0$, has a convergent subsequence in X.

In the point set topology literature, Property (F) corresponds to the property that the restriction of g to any closed bounded set of X is proper at 0. If X is finite dimensional, any map $g : X \to Y$ has Property (F). When g is linear, g has Property (F) if, and only if, g is a semi-Fredholm linear operator with index in $\mathbb{Z} \cup \{-\infty\}$, which means that g has a finite dimensional kernel and a closed range, see for instance [Hör94, Proposition 19.1.3] or [Kat95, Chapter IV, Theorems 5.10 and 5.11]. In the sequel, Id denotes the identity map over any set.

Lemma 5.2 ([AGN11, Lemma 4.1])**.** *If D is a subset of a Banach space $(X, \|\cdot\|)$, and if $h : D \to X$ is a map sending bounded sets to bounded sets and such that $\nu_D(h) < 1$, then* $\mathrm{Id} - h$ *has Property* (F)*.*

In the particular case where the homogeneous generalized measure of noncompactness ν is equal to α, and where h is continuous, Lemma 5.2 is a consequence of Corollary 2 of [Nus71], which says more generally that the restriction of $\mathrm{Id} - h$ to any closed bounded set of X is proper (at any point).

Proposition 5.3 ([AGN11, Proposition 4.2])**.** *If C is a cone of a Banach space $(X, \|\cdot\|)$, if $h : C \to C$ is homogeneous and uniformly continuous on bounded sets*

of C, and if either $\rho_C(h) < 1$ or $\tilde{r}_C(h) < 1$, then $\mathrm{Id} - h$ *has Property* (F) *on C. Moreover, when $\tilde{r}_C(h) < 1$, 0 is the unique fixed point of h in C.*

6. General uniqueness and convergence results

In this section, we establish the main results of this paper, concerning the uniqueness of the fixed point of a semidifferentiable maps and the convergence of the orbits to it. The results are already useful in the finite dimensional case, hence, the reader might wish at the first reading to ignore the technical compactness assumptions regarding Property (F), which are trivially satisfied in finite dimension.

6.1. Uniqueness of the fixed point.

We study the uniqueness of the fixed point v of a nonexpansive map defined on a metric space (V, d). We shall need the following assumptions:

(B1) There exists a Banach space $(E_v, \|\cdot\|_v)$, such that V is an open subset of E_v and that for any U contained in V U is open in the d-metric topology iff it is open in the norm topology of E_v. In addition, we require that
$$d(x,y) \sim \|x-y\|_v \text{ when } x,y \to v, \quad x,y \in V \quad , \tag{6.1}$$
where by (6.1), we mean that for all $\lambda > 1$, there exists a neighborhood U of v in V such that
$$\frac{1}{\lambda}\|x-y\|_v \leqslant d(x,y) \leqslant \lambda \|x-y\|_v \quad \forall x,y \in U \quad ; \tag{6.2}$$

(B2) The balls of (V, d) are convex in E_v;

(B3) For all $w \in V$, and $s \in (0,1)$,
$$\Gamma_s := \{z \in V \mid d(z,v) = sd(v,w),\ d(z,w) = (1-s)d(v,w)\} \neq \emptyset \quad . \tag{6.3}$$

To interpret assumption (B3), let us recall that a metric space (V, d) is *strongly metrically convex* or is a *geodesic space* if for all $x, y \in V$, there exists a (minimal) *geodesic* from x to y, that is, a continuous path, $z : [0,1] \to V$ such that $z(0) = x$, $z(1) = y$, and $d(z(s), z(t)) = |s-t|d(x,y)$ for all $s,t \in [0,1]$. Assumption (B3) holds when (V, d) is strongly metrically convex. Conversely, if (V, d) is a complete metric space and if Assumption (B3) holds for all $v \in V$, or more generally, if V is *metrically convex*, which means that for all $x, y \in V$, there exists $w \in V$ such that $x \neq w$, $y \neq w$ and $d(x,y) = d(x,w) + d(w,y)$, then a theorem of K. Menger (see [Blu53, Th. 14.1, page 41]) asserts that (V, d) is strongly metrically convex.

Assumptions (B1), (B2) and (B3) are trivially satisfied when $V = E$ and $d(x,y) = \|x-y\|$ for some Banach space $(E, \|\cdot\|)$.

Almost all the uniqueness results of this paper will be applications of the following general theorem.

Theorem 6.1. *Let (V, d) be a complete metric space satisfying* (B1)–(B3)*, G be an open subset of V, $f : G \to V$ be a nonexpansive map and $v \in G$ be a fixed point of f: $f(v) = v$. Make the following assumptions:*

(A1) *$f : G \to E_v$ is semidifferentiable at v;*

(A2) *The map* $\mathrm{Id} - f'_v : E_v \to E_v$ *has Property* (F)*;*

(A3) *The fixed point of $f'_v : E_v \to E_v$ is unique: $f'_v(x) = x$, $x \in E_v \Rightarrow x = 0$.*

Then, the fixed point of f in G is unique: $f(w) = w$, $w \in G \Rightarrow w = v$.

Theorem 6.1 can be remembered by saying that *the uniqueness of the fixed point of f'_v implies the uniqueness of the fixed point of f*, under assumptions which, as we will see in Sections 7 and 9, are fulfilled in many situations relative to cones. In many applications, the map f'_v is much "simpler" than f, and the uniqueness of the fixed point, (A3), can be proved by direct algebraic or combinatorial means. An example of such a situation will be given in §10.2.

Remark 6.2. Assumption (B1) implies that there exists $\varepsilon > 0$, such that $U = B(v, \varepsilon)$ satisfies (6.2), where in any metric space (X, d), we denote by $B(x, \rho) := \{y \in X \mid d(y, x) \leqslant \rho\}$ the closed ball of X with center $x \in X$ and radius $\rho \geqslant 0$. Thus, the topologies defined on $B(v, \varepsilon)$ by d and $\|\cdot\|_v$ are the same. This implies, in particular, that the completeness of $B(v, \varepsilon)$ for d is equivalent to its completeness for $\|\cdot\|_v$. But for a normed vector space, the completeness of any closed ball of positive radius is equivalent to the completeness of the space. Therefore, the completeness of $B(v, \varepsilon)$ for d is equivalent to the completeness of $(E_v, \|\cdot\|_v)$, and in the assumptions of Theorem 6.1, one may either omit the completeness of (V, d) or that of $(E_v, \|\cdot\|_v)$.

Remark 6.3. In Theorem 6.1, Assumption (A2) may be restrictive. Consider $E = \mathscr{C}_0([0, 1])$ the Banach space of continuous functions x from $[0, 1]$ to $\mathbb{R}$ such that $x(0) = 0$, endowed with the sup-norm, and $f : E \to E$ defined by $f(x)(t) = x(\frac{t}{2}) \vee (x(t) - 1)$. We get that f is nonexpansive and that $v \equiv 0$ is the unique fixed point of f, that f is differentiable at v with semiderivative $f'_v(x)(t) = x(\frac{t}{2})$, that f'_v has 0 as a unique fixed point, whereas $\mathrm{Id} - f'_v$ does not have Property (F). Indeed, $x_n(t) = t^{1/n}$ is such that $x_n - f'_v(x_n)$ tends to 0 when n goes to infinity, whereas $\langle x_n \mid n \geqslant 1\rangle$ has no convergent subsequence (see [Bon58]).

As a corollary of Lemma 3.2, we obtain the following proposition, which gives in particular a sufficient condition for Assumption (A1) of Theorem 6.1 to hold.

Proposition 6.4. *Let (V, d) be a metric space satisfying Assumption* (B1)*, let G be an open subset of V, let $v \in V$, and let $f : G \to V$ be a nonexpansive map that has directional derivatives at v with respect to all $x \in E_v$. Then, $f'_v : E_v \to E_v$ is nonexpansive, and if E_v is finite dimensional, f is semidifferentiable at v.*

Proof. We first prove that the map f'_v (defined by (3.2)) is nonexpansive with respect to $\|\cdot\|_v$. By taking $X = C = E_v$ and a neighborhood U of v together with $\lambda > 1$ as in (6.2), so that f is Lipschitz of constant $M = \lambda^2$ in U, we get from the proof of Lemma 3.2 that f'_v is λ^2-Lipschitz $(E_v, \|\cdot\|_v) \to (E_v, \|\cdot\|_v)$. Since this holds for all $\lambda > 1$, f'_v is nonexpansive $(E_v, \|\cdot\|_v) \to (E_v, \|\cdot\|_v)$. The remaining part of the proposition follows from Lemma 3.2. ☐

The proof of Theorem 6.1 relies on the following general local uniqueness result, which does not require f to be nonexpansive.

Lemma 6.5. *Let $(E, \|\cdot\|)$ be a normed vector space, f be a map from a subset $G \subset E$ to E, and v be a fixed point of f belonging to the interior of G. Make the following assumptions:*

(A1) *$f : G \to E$ is semidifferentiable at v.*
(A2) *The map $\mathrm{Id} - f'_v : E \to E$ has Property* (F)*;*
(A3) *The fixed point of $f'_v : E \to E$ is unique: $f'_v(x) = x$, $x \in E \Rightarrow x = 0$.*

Then, there does not exist a sequence $\langle v_n \in E \setminus \{v\} \mid n \geqslant 1\rangle$ *such that*

$$\lim_{n\to\infty} \|v_n - v\| = 0, \text{ and } \lim_{n\to\infty} \frac{\|f(v_n) - v_n\|}{\|v_n - v\|} = 0 \ . \tag{6.4}$$

In particular v *is isolated in the set of fixed points of* f*.*

Proof. Let $\langle v_n \in E \mid n \geqslant 1\rangle$ be as in (6.4). Writing $v_n := v + \varepsilon_n x_n$ with $\varepsilon_n = \|v_n - v\|$, we get $\lim_{n\to\infty} \varepsilon_n = 0$ and $\|x_n\| = 1$. Since the sequence $\langle x_n \mid n \geqslant 1\rangle$ is bounded, it follows from (3.3a) and (3.3b), which are satisfied thanks to (A1), that

$$\lim_{n\to\infty} \left\| \frac{f(v_n) - f(v)}{\varepsilon_n} - f'_v(x_n) \right\| = 0. \tag{6.5}$$

Recalling that $f(v) = v$ and using the second equation in (6.4) together with (6.5), we get

$$\lim_{n\to\infty} \|x_n - f'_v(x_n)\| = 0 \ . \tag{6.6}$$

Since the sequence x_n is bounded, Assumption (A2) implies that x_n has a subsequence converging to a point $x \in E$. Using the continuity of the semiderivative f'_v, we obtain, from (6.6), $x - f'_v(x) = 0$. By Assumption (A3), this implies that $x = 0$, a contradiction with $\|x\| = 1$. □

Remark 6.6. Let us give an example where all the assumptions of Lemma 6.5 are fulfilled except Assumption (A2) and where the fixed point v is not isolated. Consider $E = \mathscr{C}_0([0,1])$ the Banach space of continuous functions x from $[0,1]$ to $\mathbb{R}$ such that $x(0) = 0$, endowed with the sup-norm, and $f : E \to E$ defined by $f(x)(t) = x(\frac{t}{2}) + x(1)x(t)$. Note that f is not nonexpansive on E. For all $\gamma \in [0,1]$, let $x_\gamma \in E$ be such that $x_\gamma(t) = (1 - (\frac{1}{2})^\gamma)t^\gamma$ for all $t \in [0,1]$. Then, x_γ is a fixed point of f for all $\gamma \in [0,1]$, $v := x_0 \equiv 0$ and x_γ tends to v when γ tends to 0. Hence, v is a non isolated fixed point of f. The map f is differentiable at v with semiderivative $f'_v(x)(t) = x(\frac{t}{2})$, and we already pointed out above that f'_v has 0 as a unique fixed point, whereas $\mathrm{Id} - f'_v$ does not have Property (F).

Proof of Theorem 6.1. Suppose, by way of contradiction, that there exists $w \in G$ such that $f(w) = w$, $w \neq v$. Let $R = d(v,w) > 0$, and for all $s \in (0,1)$, consider the set Γ_s defined by (6.3). Since G is open, $v \in G$ and $\Gamma_s \subset B(v, Rs)$, there exists $\bar{s} \in (0,1]$ such that $\Gamma_s \subset G$, for all $s \leqslant \bar{s}$. By Assumption (B3), for all $s \in (0,1)$, there exists $z_s \in \Gamma_s$.

Also, we have:

$$\Gamma_s = \{z \in V \mid \ d(z,v) \leqslant sd(v,w),\ d(z,w) \leqslant (1-s)d(v,w)\} \ . \tag{6.7}$$

Indeed, if $z \in V$ is such that $d(z,v) \leqslant sd(v,w)$ and $d(z,w) \leqslant (1-s)d(v,w)$, we get

$$d(v,w) \leqslant d(z,v) + d(z,w) \leqslant sd(v,w) + (1-s)d(v,w) = d(v,w) \ ,$$

hence $d(z,v) = sd(v,w)$ and $d(z,w) = (1-s)d(v,w)$. From (6.7) and Assumption (B2), we deduce that Γ_s is convex. Because f is nonexpansive with respect to d, $f(v) = v$, $f(w) = w$, and Γ_s is given by (6.7), we have that $f(\Gamma_s) \subset \Gamma_s$ (for $s \leqslant \bar{s}$). This property, together with the convexity of Γ_s, allows us to consider, for $t \in [0,1]$ and $s \in [0,\bar{s}]$, the map $f_{s,t} : \Gamma_s \to \Gamma_s$,

$$f_{s,t}(x) = (1-t)f(x) + tz_s \ . \tag{6.8}$$

By Assumption (B1), for all $\lambda > 1$, there exists $0 < s_\lambda \leqslant \bar{s}$ such that:

$$\frac{1}{\lambda}\|x-y\|_v \leqslant d(x,y) \leqslant \lambda\|x-y\|_v \quad \forall x,y \in B(v,sR),\ 0<s\leqslant s_\lambda \ . \tag{6.9}$$

Since $\Gamma_s \subset B(v,sR)$, and f is nonexpansive with respect to d, we get (using (6.9)) that for all $x,y \in \Gamma_s$ and $0 < s \leqslant s_\lambda$,

$$\begin{aligned} d(f_{s,t}(x), f_{s,t}(y)) &\leqslant \lambda\|f_{s,t}(x) - f_{s,t}(y)\|_v \\ &= \lambda(1-t)\|f(x)-f(y)\|_v \\ &\leqslant \lambda^2(1-t)d(f(x),f(y)) \\ &\leqslant \lambda^2(1-t)d(x,y) \ . \end{aligned}$$

Considering

$$t_\lambda = \min(2(1-\frac{1}{\lambda^2}),1) \in (0,1] \ ,$$

we get that

$$c_\lambda := \lambda^2(1-t_\lambda) < 1 \text{ and } \lim_{\lambda\to 1^+} t_\lambda = 0 \ . \tag{6.10}$$

Hence, for all $0 < s \leqslant s_\lambda$ and $t_\lambda \leqslant t \leqslant 1$, $f_{s,t}$ is a contraction mapping in (Γ_s, d), with contraction factor c_λ. Since Γ_s is closed in V, (Γ_s, d) is a complete metric space, hence the contraction mapping principle implies that $f_{s,t}$ has a unique fixed point $v_{s,t} \in \Gamma_s$. By definition, $v_{s,t}$ satisfies:

$$f(v_{s,t}) - v_{s,t} = t(f(v_{s,t}) - z_s) \ , \tag{6.11}$$

which leads to

$$\begin{aligned} \|f(v_{s,t}) - v_{s,t}\|_v &= t\|f(v_{s,t}) - z_s\|_v \\ &\leqslant t\lambda d(f(v_{s,t}), z_s) \\ &\leqslant t\lambda(d(f(v_{s,t}),v) + d(v,z_s)) \\ &\leqslant 2t\lambda Rs \ , \end{aligned} \tag{6.12}$$

since $f(v_{s,t})$ and $z_s \in \Gamma_s$.

Since $v_{s,t} \in \Gamma_s$ and Γ_s is given by (6.3), we get, using (6.9):

$$\frac{Rs}{\lambda} = \frac{d(v, v_{s,t})}{\lambda} \leqslant \|v_{s,t} - v\|_v \leqslant \lambda d(v, v_{s,t}) = \lambda Rs \ . \tag{6.13}$$

The first inequality in (6.13) together with (6.12) yield:

$$\frac{\|f(v_{s,t}) - v_{s,t}\|_v}{\|v_{s,t} - v\|_v} \leqslant 2t\lambda^2 \ . \tag{6.14}$$

Let us choose a sequence $\langle \lambda_n > 1 \mid n \geqslant 1\rangle$, such that $\lambda_n \to_{n\to\infty} 1$, together with a sequence $\langle s_n > 0 \mid n \geqslant 1\rangle$, such that $s_n \leqslant s_{\lambda_n}$ and $s_n \to 0$. We set $t_n := t_{\lambda_n}$ and $v_n := v_{s_n,t_n}$, to simplify the notation. By (6.10), $t_n \to 0$. Using (6.13) and (6.14), we get

$$\lim_{n\to\infty} \|v_n - v\|_v = 0, \quad \lim_{n\to\infty} \frac{\|f(v_n) - v_n\|_v}{\|v_n - v\|_v} = 0 \ . \tag{6.15}$$

Taking $E = E_v$, $\|\cdot\| = \|\cdot\|_v$, we see that Assumptions (A1)–(A3) of Lemma 6.5 are satisfied due to Assumptions (A1)–(A3) of Theorem 6.1. Thus, the conclusion of Lemma 6.5 contradicts (6.15). □

Remark 6.7. The introduction of $f_{s,t}$ as in (6.8) and the derivation of its properties is closely related to Lemma 2.1 on page 45 of [Nus88].

6.2. **Geometric convergence to the fixed point.** When $\tilde{r}(f'_v) < 1$, we can prove a result more precise than Theorem 6.1: the geometric convergence of the orbits of f towards the fixed point of f.

Theorem 6.8. *Let (V,d) be a complete metric space, G be a non-empty connected open subset of V, $f: G \to G$ be a nonexpansive map and $v \in G$ be a fixed point of f: $f(v) = v$. Assume that Assumptions* (B1) *and* (A1) *of Theorem 6.1 hold and that $f'_v : E_v \to E_v$ satisfies $\tilde{r}(f'_v) < 1$, where $\tilde{r} = \tilde{r}_{E_v}$ is defined with respect to the $\|\cdot\|_v$ norm. Then,*

$$\limsup_{k\to\infty} d(f^k(x), v)^{1/k} \leqslant \tilde{r}(f'_v) \quad \forall x \in G \ .$$

In particular, the fixed point of f in G is unique.

Remark 6.9. Under the assumptions of Theorem 6.8, f'_v is nonexpansive (by Proposition 6.4), hence $\mathrm{Id} - f'_v$ has Property (F) and, by Proposition 5.3, the fixed point of f'_v is unique. This shows that Assumptions (A2) and (A3) of Theorem 6.1 are satisfied. Therefore, Theorem 6.1 shows the uniqueness of the fixed point of f without the connectedness of G, as soon as Assumptions (B2) and (B3) on (V,d) are satisfied.

To show Theorem 6.8, we first prove that there is a neighborhood of v in which all orbits of f converge geometrically to v:

Lemma 6.10. *Let V, d, G, f and v be as in Theorem 6.8, and $(E_v, \|\cdot\|_v)$ be as in* (B1). *For all $1 > \mu > \tilde{r}(f'_v)$, there exists $\eta > 0$ and $m > 0$ such that*

$$d(f^m(x), v) \leqslant \mu^m d(x, v) \quad \forall x \in B(v, \eta) \ . \tag{6.16}$$

Proof. Since $\mu > \tilde{r}(f'_v)$, and $\tilde{r}(f'_v)$ is defined in $(E_v, \|\cdot\|_v)$, we can chose m such that $\|(f'_v)^m\|_v < \mu^m$. The chain rule for semidifferentiable maps (Lemma 3.4), together with $f(v) = v$, show that f^m is semidifferentiable at v, with $(f^m)'_v = (f'_v)^m$:

$$f^m(x) - v = f^m(x) - f^m(v) = (f'_v)^m(x - v) + o(\|x - v\|_v) \ . \tag{6.17}$$

Hence,

$$\|f^m(x) - v\|_v \leqslant \|(f'_v)^m\|_v \|x - v\|_v + o(\|x - v\|_v) \ ,$$

and using (6.1), we get that there is a ball $B(v, \eta)$, with $\eta > 0$, in which (6.16) holds. $\square$

Proof of Theorem 6.8. We get from (6.16):

$$d(f^{mk}(x), v) \leqslant \mu^{mk} d(x, v) \quad \forall k \geqslant 0,\ x \in B(v, \eta) \ . \tag{6.18}$$

Moreover, by nonexpansiveness of f,

$$d(f^{k+1}(x), v) = d(f^{k+1}(x), f^{k+1}(v)) \leqslant d(f^k(x), f^k(v)) = d(f^k(x), v) \ . \tag{6.19}$$

Combining (6.18) and (6.19), we see that

$$\limsup_{k} d(f^k(x), v)^{1/k} \leqslant \mu \quad \forall x \in B(v, \eta) \ . \tag{6.20}$$

Let

$$\Omega = \{x \in G \mid \lim_k f^k(x) = v\} \ .$$

We claim that $\Omega = G$. By (6.20), $B(v,\eta) \subset \Omega$, hence $\Omega \neq \emptyset$. We will show that Ω is both open and closed in G. We claim that:

$$B(x,\eta/2) \cap G \subset \Omega \quad \forall x \in \Omega \ . \tag{6.21}$$

Indeed, if $x \in \Omega$, we have $d(f^n(x),v) < \eta/2$ for some n, hence, for all $y \in G$ such that $d(y,x) \leqslant \eta/2$, $d(f^n(y),v) \leqslant d(f^n(y),f^n(x)) + d(f^n(x),v) < \eta$, and by (6.20), $\lim_k f^k(y) = v$, which shows (6.21), and, a fortiori, that Ω is open. Property (6.21) also implies that Ω is closed. Indeed, if $\langle x_n \in \Omega \mid n \geqslant 1\rangle$ is a sequence converging to $x \in G$, we have $d(x_n,x) < \eta/2$ for some $n \geqslant 1$, hence, $x \in \Omega$ by (6.21). We have shown that Ω is non-empty, closed and open in G, and since G is connected, $\Omega = G$. Finally, since $\lim_k f^k(x) = v$ for all $x \in G$, and since (6.20) holds, we have $\limsup_k d(f^k(x),v)^{1/k} \leqslant \mu$, for all $x \in G$. Since this inequality holds for all $\mu > \tilde{r}(f'_v)$, we have proved (6.16). □

7. Fixed points and eigenvectors of semidifferentiable nonexpansive maps over normal cones

In this section, we derive from Theorem 6.1 some uniqueness results for fixed points or eigenvectors of maps acting on cones, as well as geometric convergence results for the orbits.

As a first application, we get the following uniqueness result for the fixed point of a nonexpansive map in the Thompson's metric.

Theorem 7.1 (Uniqueness of fixed points)**.** *Let C be a normal cone with nonempty interior in a Banach space $(X,\|\cdot\|)$. Let G be a nonempty open subset of $(X,\|\cdot\|)$ included in C and $f : G \to \operatorname{int} C$ be a nonexpansive map with respect to Thompson's metric $\bar{d}$. Let $v \in G$ be a fixed point of f: $f(v) = v$. Make the following assumptions:*

- (A1) *f is semidifferentiable at v;*
- (A2) *The map $\mathrm{Id} - f'_v : X \to X$ has Property* (F)*;*
- (A3) *The fixed point of $f'_v : X \to X$ is unique: $f'_v(x) = x$, $x \in X \Rightarrow x = 0$.*

Then, the fixed point of f in G is unique: $f(w) = w$, $w \in G \Rightarrow w = v$.

Proof. We apply Theorem 6.1. Let $V = \operatorname{int} C$ endowed with $\bar{d}$. The set G is open in $(V,\bar{d})$, f is nonexpansive with respect to $\bar{d}$ and v is a fixed point of f. Since $v \in \operatorname{int} C$, $C_v = \operatorname{int} C = V$ and by Proposition 2.1, $(V,\bar{d})$ is a complete metric space. Moreover, by Proposition 2.4, $\|\cdot\|$, $\|\cdot\|_v$ and $\bar{d}$ define the same topology on V. Since the equivalence (2.23) holds, $(V,\bar{d})$ satisfies (B1) with $E_v = X_v$ and $\|\cdot\|_v$. From the definition of $\bar{d}$ and the convexity of C, it follows that the metric space $(V,\bar{d})$ satisfies Assumption (B2). Assumption (B3) for $(V,\bar{d})$ follows from the existence of a (minimal) geodesic for $\bar{d}$ between any two points of $\operatorname{int} C$, which is proved in [Nus88, Proposition 1.12, page 34]. Then, Assumptions (A1)–(A3) of Theorem 7.1 correspond to Assumptions (A1)–(A3) of Theorem 6.1, respectively, which yields the conclusion of the theorem. □

Theorem 7.2 (Geometric convergence)**.** *Let C be a normal cone with non-empty interior in a Banach space $(X,\|\cdot\|)$, and let G be a non-empty open subset of $(X,\|\cdot\|)$ included in C. Assume that $f : G \to G$ is nonexpansive with respect to Thompson's metric $\bar{d}$. If f has a fixed point $v \in G$, and if f is semidifferentiable*

at v, with $\tilde r(f'_v)<1$, where $\tilde r$ is defined with respect to the $\|\cdot\|$ or the $\|\cdot\|_v$ norm, then the fixed point of f in G is unique. Moreover, if G is connected, we have:

$$\forall x\in G,\ \ \limsup_{k\to\infty}\bar d(f^k(x),v)^{1/k}\leqslant \tilde r(f'_v)\ \ .$$

Proof. By the proof of Theorem 7.1, $(\operatorname{int} C,d)$ satisfies (B1). By Proposition 6.4, f'_v is nonexpansive, and, a fortiori, uniformly continuous on bounded sets of C. Therefore, Proposition 5.3 implies that Assumptions (A2) and (A3) of Theorem 7.1 are satisfied. Since the other assumptions of Theorem 7.2 imply the other assumptions of Theorem 7.1, we get the first assertion of Theorem 7.2. (See also Remark 6.9.) The last one is a direct corollary of Theorem 6.8. □

Theorem 7.3 (Uniqueness of fixed points in Σ)**.** *Let C be a normal cone with nonempty interior in a Banach space $(X,\|\cdot\|)$, let $\psi\in C^*\setminus\{0\}$, and $\Sigma=\{x\in \operatorname{int} C\mid \psi(x)=1\}$. Let G be a nonempty relatively open subset of Σ and $g:G\to\Sigma$ be a nonexpansive map with respect to Hilbert's projective metric d or Thompson's metric $\bar d$. Let $v\in G$ be a fixed point of g: $g(v)=v$. Make the following assumptions:*

(A1) *g is semidifferentiable at v with respect to $\psi^{-1}(0)$;*
(A2) *The map $\mathrm{Id}-g'_v:\psi^{-1}(0)\to\psi^{-1}(0)$ has Property* (F)*;*
(A3) *The fixed point of $g'_v:\psi^{-1}(0)\to\psi^{-1}(0)$ is unique: $g'_v(x)=x$, $x\in\psi^{-1}(0)\Rightarrow x=0$.*

Then, the fixed point of g in G is unique: $g(w)=w$, $w\in G\Rightarrow w=v$.

Proof. We apply Theorem 6.1. Let $V=-v+\Sigma$. We consider on V the two following metrics $d_v(x,y):=d(v+x,v+y)$ and $\bar d_v(x,y):=\bar d(v+x,v+y)$. Let $G_v=-v+G$ and consider the map $f:G_v\to V$, $f(x)=g(v+x)-v$. If g is nonexpansive with respect to d (resp. $\bar d$), then f is nonexpansive with respect to d_v (resp. $\bar d_v$), and satisfies $f(0)=0$.

We know, by Proposition 2.2, that (Σ,d) and $(\Sigma,\bar d)$, hence (V,d_v) and $(V,\bar d_v)$ are complete metric spaces. Since $v\in\operatorname{int}C$, Proposition 2.4, together with (2.4), (2.8), and (2.9) show that $\|\cdot\|$, $\|\cdot\|_v$, ω_v, $\bar d$, and d define the same topology on Σ. Hence, $\|\cdot\|$, $\|\cdot\|_v$, ω_v, d_v and $\bar d_v$ define the same topology on V. Since the equivalence (2.23) (resp. (2.24)) holds, $(V,\bar d_v)$ (resp. (V,d_v)) satisfies (B1) with $E_0=\psi^{-1}(0)$ and $\|\cdot\|_0=\omega_v$ (resp. $\|\cdot\|_0=\|\cdot\|_v$). Moreover, the set G_v is open in (V,d_v) and $(V,\bar d_v)$. From the definition of d, $\bar d$ and the convexity of C, it follows that the metric spaces (V,d_v) and $(V,\bar d_v)$ both satisfy Assumption (B2). Assumption (B3) for (V,d_v) (resp. $(V,\bar d_v)$) follows from the existence of a (minimal) geodesic for d (resp. $\bar d$) between any two points of Σ_v, which is proved in [Nus88, Proposition 1.9, page 25] (resp. [Nus88, Proposition 1.12, page 34]). Then, Assumptions (A1)–(A3) of Theorem 7.3 correspond to Assumptions (A1)–(A3) of Theorem 6.1, which yields the conclusion of Theorem 7.3. □

Remark 7.4. The proof of Theorem 7.1 remains valid if one replaces $\operatorname{int}C$ by the cone C_u for some $u\in C\setminus\{0\}$ where C is still a normal cone, while assuming that G is an open subset of $(C_u,\bar d)$, and replacing $\|\cdot\|$ by $\|\cdot\|_u$. Similarly, the statements of Theorem 7.3, and also of Theorem 7.5 and Corollaries 9.1, 9.3 and 7.7 below, have obvious extensions applying to C_u or Σ_u.

In order to obtain a uniqueness result for an eigenvector of f, we shall apply Theorem 7.3 to the map g introduced in Lemma 4.4. Natural assumptions on f

which ensure the nonexpansiveness of g together with Assumptions (A1)–(A3) in Theorem 7.3, are captured in the following result.

Theorem 7.5 (Uniqueness of eigenvectors in Σ)**.** *Let C be a normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$, let $\psi \in C^* \setminus \{0\}$, and denote $\Sigma = \{x \in \operatorname{int} C \mid \psi(x) = 1\}$. Let G be a nonempty open subset of $(X, \|\cdot\|)$ included in C and $f : G \to \operatorname{int} C$ be a map such that $f|_{G\cap\Sigma}$ is nonexpansive with respect to Hilbert's projective metric d. Assume that $v \in G \cap \Sigma$ is a fixed point of f: $f(v) = v$. Make the following assumptions:*

(A1) *f is semidifferentiable at v;*
(A2) *The map $(\mathrm{Id} - f'_v)|_{\psi^{-1}(0)} : \psi^{-1}(0) \to X$ has Property* (F)*;*
(A3) *The fixed point of f'_v in $\psi^{-1}(0)$ is unique: $f'_v(x) = x$, $x \in \psi^{-1}(0) \Rightarrow x = 0$;*
(A4) *f'_v is order preserving;*
(A5) *f'_v is additively subhomogeneous with respect to v.*

Then, the eigenvector of f in $G \cap \Sigma$ is unique: $\exists \lambda > 0$, $f(w) = \lambda w$, $w \in G \cap \Sigma \Rightarrow w = v$.

Proof. Let $\tilde{f}$ be defined by (4.4) and $g = \tilde{f}|_{G\cap\Sigma} : G \cap \Sigma \to \Sigma$. We shall prove that g satisfies the assumptions of Theorem 7.3 with G replaced by $G \cap \Sigma$. First, since $f|_{G\cap\Sigma}$ is nonexpansive with respect to d, so is g (by Lemma 4.4). Since, $f(v) = v$ and $v \in G \cap \Sigma$, $g(v) = v$. From Assumption (A1) of Theorem 7.5 and Lemma 4.4, we get Assumption (A1) of Theorem 7.3. It remains to check Assumptions (A2) and (A3) of Theorem 7.3.

Let us first show that

$$\|x - f'_v(x)\|_v \leqslant 2\,\|x - g'_v(x)\|_v \quad \forall x \in \psi^{-1}(0)\ . \tag{7.1}$$

Since $f'_v(0) = 0$ and, by (A4) and (A5), f'_v is order preserving and additively subhomogeneous with respect to v, $h = f'_v$ satisfies the assumptions of Lemma 4.5 with $S \ni 0$. Hence, from (4.7) and (4.8), we obtain, for all $x \in X$,

$$m(x - f'_v(x) \,/\, v) \leqslant m(x \,/\, v) \vee 0 \tag{7.2}$$

$$M(x - f'_v(x) \,/\, v) \geqslant M(x \,/\, v) \wedge 0\ . \tag{7.3}$$

Since $\psi \in C^*$, $v \in \Sigma$ and $m(x \,/\, v)v \leqslant x \leqslant M(x \,/\, v)v$, we get $m(x \,/\, v) \leqslant \psi(x) \leqslant M(x \,/\, v)$. Hence, for all $x \in \psi^{-1}(0)$, $m(x \,/\, v) \leqslant 0 \leqslant M(x \,/\, v)$ which with (7.2) and (7.3) leads to

$$m(x - f'_v(x) \,/\, v) \leqslant 0 \leqslant M(x - f'_v(x) \,/\, v)\ . \tag{7.4}$$

Denote $y = x - g'_v(x)$. By (4.5) and $g'_v = \tilde{f}'_v|_{\psi^{-1}(0)}$, we get

$$x - f'_v(x) = y - \psi(f'_v(x))v\ , \tag{7.5}$$

thus $m(x - f'_v(x) \,/\, v) = m(y \,/\, v) - \psi(f'_v(x))$ and $M(x - f'_v(x) \,/\, v) = M(y \,/\, v) - \psi(f'_v(x))$. Using (7.4), we get $m(y \,/\, v) \leqslant \psi(f'_v(x)) \leqslant M(y \,/\, v)$, hence

$$|\psi(f'_v(x))| \leqslant \|y\|_v\ . \tag{7.6}$$

Gathering (7.5) and (7.6), we get

$$\|x - f'_v(x)\|_v \leqslant \|y\|_v + |\psi(f'_v(x))| \leqslant 2\|y\|_v\ ,$$

which shows (7.1).

Using (7.1), we get that, if $g'_v(x) = x$ and $x \in \psi^{-1}(0)$, then $f'_v(x) = x$, whence by Assumption (A3) in Theorem 7.5, $x = 0$. This shows Assumption (A3) in Theorem 7.3. Let now $\langle x_j \mid j \geqslant 1\rangle$ be a bounded sequence in $\psi^{-1}(0)$ such that

$x_j - g'_v(x_j) \to_{j\to\infty} 0$. By (7.1), this implies that $x_j - f'_v(x_j) \to_{j\to\infty} 0$, whence, by Assumption (A2) of Theorem 7.5, $\langle x_j \mid j \geqslant 1\rangle$ admits a convergent subsequence. This shows Assumption (A2) of Theorem 7.3, and completes the verification of the assumptions of this theorem.

If $w \in G \cap \Sigma$ satisfies $f(w) = \lambda w$ for some $\lambda > 0$, then $g(w) = w$, hence, by Theorem 7.3, $w = v$. □

Remark 7.6. As a consequence of Lemma 4.2, Assumption (A4) of Theorem 7.5 is fulfilled as soon as f is order preserving in a neighborhood of v, and Assumption (A5) is fulfilled as soon as f is subhomogeneous in a neighborhood of v. These properties also imply, by Lemma 4.1, that $f|_{G\cap\Sigma}$ is nonexpansive with respect to Hilbert's projective metric d. They will be used in particular in Corollary 7.7. Another way to ensure (A5) is given by Lemma 4.3. In particular, (A5) is fulfilled when f'_v is linear (that is f is differentiable at v) and $f'_v(v) \leqslant v$. Moreover, by Lemma 4.2, (ii), Assumption (A5) is fulfilled when f is convex and $f'_v(v) \leqslant v$.

Corollary 7.7 (Uniqueness of eigenvectors)**.** *Let C be a normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let $f : \operatorname{int} C \to \operatorname{int} C$ be homogeneous and order-preserving. Assume that $S = \{x \in \operatorname{int} C \mid f(x) = x\}$ is nonempty, and that $v \in S$. Make the following assumptions:*

- (A1) *f is semidifferentiable at v;*
- (A2) *The map $\mathrm{Id} - f'_v : X \to X$ has Property* (F)*;*
- (A3) *if $f'_v(x) = x$ for some $x \in X$, then $x \in \{\lambda v \mid \lambda \in \mathbb{R}\}$;*

Then, $S = \{\lambda v \mid \lambda > 0\}$.

Proof. Let us first check that f satisfies the assumptions of Theorem 7.5. Consider $G = \operatorname{int} C$ which is clearly open. Since $v \in C \setminus \{0\}$, one can choose $\psi \in C^* \setminus \{0\}$ such that $\psi(v) = 1$. In that case, $v \in \Sigma = G \cap \Sigma$ and $f(v) = v$. Since f is order preserving and homogeneous, $f|_\Sigma$ is nonexpansive with respect to d. Assumption (A1) of Corollary 7.7 corresponds to Assumption (A1) of Theorem 7.5. Assumption (A2) of Corollary 7.7 implies Assumption (A2) of Theorem 7.5. In order to show Assumption (A3) of Theorem 7.5, let us consider $x \in X$ such that $f'_v(x) = x$ and $\psi(x) = 0$. By Assumption (A3) of Corollary 7.7, $x = \lambda v$ for some $\lambda \in \mathbb{R}$, and since $\psi(x) = 0$, we get $\lambda = 0$, thus $x = 0$, which shows Assumption (A3) of Theorem 7.5. Assumptions (A4) and (A5) of Theorem 7.5 are deduced from Lemma 4.2, (i) and (iii) respectively, using the fact that f is order preserving and homogeneous. This completes the proof of the assumptions of Theorem 7.5.

Let $x \in S$, then $\lambda = \psi(x) > 0$, and, since f is homogeneous, $y = \frac{x}{\lambda}$ satisfies $f(y) = y$ and $y \in \Sigma$. From Theorem 7.5, this implies $y = v$, hence $x = \lambda v$. Conversely, if $x = \lambda v$ for some $\lambda > 0$, then $x \in S$, since f is homogeneous. □

If f is homogeneous, $f(v) = v$ and f satisfies Assumption (A1) of Corollary 7.7, we have trivially $f'_v(\lambda v) = \lambda v$, for all $\lambda \in \mathbb{R}$. Therefore, Assumption (A3) of Corollary 7.7 can be thought of as a uniqueness assumption for the eigenvector of f'_v, and Corollary 7.7 states in essence that the uniqueness of the eigenvector of f'_v implies the uniqueness of the eigenvector of f.

Let us state a corollary of Theorem 6.8 in the framework of Corollary 7.7. If X is a vector space endowed with a seminorm ω and h is a homogeneous self-map of

a cone C of X, we generalize (3.7) and (5.1) by setting:

$$\omega_C(h) = \sup_{\substack{x\in C\\ \omega(x)\neq 0}} \frac{\omega(h(x))}{\omega(x)} = \sup_{\substack{x\in C\\ \omega(x)=1}} \omega(h(x)) \in [0,+\infty] \tag{7.7}$$

and

$$\tilde{r}_C(h) = \lim_{k\to\infty} \omega_C(h^k)^{1/k}. \tag{7.8}$$

Again, when $C = X$, we omit C in these notations.

Theorem 7.8. *Let C be a normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let $f : \operatorname{int} C \to \operatorname{int} C$ be homogeneous and order-preserving. Assume that $S = \{x \in \operatorname{int} C \mid f(x) = x\}$ is nonempty, and that $v \in S$. Assume that f is semidifferentiable at v with $\tilde{r}(f'_v) < 1$, where $\tilde{r}$ is defined as in (7.7,7.8) with respect to the seminorm ω_v. Then, for all $x \in \operatorname{int} C$,*

$$\limsup_{k\to\infty} d(f^k(x), v)^{1/k} \leqslant \tilde{r}(f'_v) \ , \tag{7.9}$$

where d is the Hilbert's projective metric, and there is a scalar $\lambda > 0$ (depending on x) such that

$$\limsup_{k\to\infty} \bar{d}(f^k(x), \lambda v)^{1/k} \leqslant \tilde{r}(f'_v) \ , \tag{7.10}$$

where $\bar{d}$ is the Thompson's metric.

Proof. Consider as in the proof of Corollary 7.7, $G = \operatorname{int} C$ and $\psi \in C^*\backslash\{0\}$ such that $\psi(v) = 1$. Denote again $\Sigma = \{x \in \operatorname{int} C \mid \psi(x) = 1\}$. Since f is homogeneous, the functions $\tilde{f}$ of (4.4) and $g = \tilde{f}\mid_\Sigma$ satisfy :

$$f^k(x) = \psi(f^k(x))\tilde{f}^k(x) \text{ and } \tilde{f}^k(x) = g^k\left(\frac{x}{\psi(x)}\right) \ .$$

Hence

$$d(f^k(x), v) = d\left(g^k\left(\frac{x}{\psi(x)}\right), v\right). \tag{7.11}$$

Moreover, by Lemma 4.2,(iii), f'_v is additively homogeneous with respect to v, and using (4.5), we get after an immediate induction on k:

$$(\tilde{f}'_v)^k(x) = (f'_v)^k(x) - \psi((f'_v)^k(x))v = (g'_v)^k(x - \psi(x)v) \ ,$$

hence

$$\omega_v((f'_v)^k(x)) = \omega_v((\tilde{f}'_v)^k(x)) = \omega_v((g'_v)^k(x - \psi(x)v)) \ .$$

Therefore,

$$\omega((f'_v)^k) = \sup_{\substack{x\in X\\ \omega_v(x)\neq 0}} \frac{\omega_v((g'_v)^k(x-\psi(x)v))}{\omega_v(x-\psi(x)v)} = \sup_{\substack{x\in\psi^{-1}(0)\\ \omega_v(x)\neq 0}} \frac{\omega_v((g'_v)^k(x))}{\omega_v(x)} = \omega((g'_v)^k)$$

which shows that

$$\tilde{r}(f'_v) = \tilde{r}(g'_v) \ . \tag{7.12}$$

From (7.11) and (7.12), (7.9) is equivalent to

$$\forall x \in \Sigma, \qquad \limsup_{k\to\infty} d(g^k(x), v)^{1/k} \leqslant \tilde{r}(g'_v) \ . \tag{7.13}$$

Since ω_v is a norm on $\psi^{-1}(0)$ and $g : \Sigma \to \Sigma$ is nonexpansive with respect to d (by Lemma 4.4), (7.13) is obtained by applying Theorem 6.8, using the same transformations as in the first paragraph of the proof of Theorem 7.3.

We now prove (7.10). By homogeneity of f, it is enough to consider the case when $x \in \Sigma$. Then, since d and $\bar{d}$ are equivalent on Σ, we get by (7.13) and (7.12),

$$\limsup_{k\to\infty} \bar{d}(g^k(x), v)^{1/k} \leqslant \tilde{r}(g'_v) = \tilde{r}(f'_v) < 1 \ . \tag{7.14}$$

To derive (7.10) from (7.14), we write $f^k(x)$ as a function of the orbit of x under g,

$$f^k(x) = \psi(f \circ g^{k-1}(x)) \cdots \psi(f \circ g^0(x)) g^k(x) \tag{7.15}$$

(this formula is readily checked by induction on $k \geqslant 1$, using the homogeneity of f and the definition of g; this is an instance of the well known "1-cocycle" representation of iterates of homogeneous maps, see e.g. [Fur63]). We still denote by $\bar{d}$ the Thompson's metric on the open cone of strictly positive real numbers:

$$\bar{d}(\nu, \mu) = |\log \nu - \log \mu|, \qquad \forall \nu, \mu > 0 \ .$$

Since f is order preserving and homogeneous, and since $x \leqslant y \implies \psi(x) \leqslant \psi(y)$, and ψ is linear, we have:

$$\forall x, y \in \operatorname{int} C, \qquad \bar{d}(\psi \circ f(x), \psi \circ f(y)) \leqslant \bar{d}(x, y) \ . \tag{7.16}$$

(This follows from the standard argument of the proof of Lemma 4.1: we have $\exp(-\bar{d}(x,y))y \leqslant x \leqslant \exp(\bar{d}(x,y))y$, by definition of Thompson's metric, and applying $\psi \circ f$, we get (7.16).) Let $\mu_k = \psi(f \circ g^k(x))$. Since $f(v) = v$ and $\psi(v) = 1$, it follows from (7.16) that

$$|\log \mu_k| = \bar{d}(\mu_k, 1) = \bar{d}(\psi \circ f(g^k(x)), \psi \circ f(v)) \leqslant \bar{d}(g^k(x), v) \ .$$

Using (7.14), we deduce:

$$\limsup_{k\to\infty} |\log \mu_k|^{1/k} \leqslant \tilde{r}(g'_v) < 1 \ . \tag{7.17}$$

Hence, the series $\sum_{k\geqslant 0} \log \mu_k$ is absolutely convergent, which implies that the infinite product $\lambda = \prod_{k\geqslant 0} \mu_k$ is convergent (with $\infty > \lambda > 0$). We get from (7.15):

$$\begin{aligned}
\bar{d}(f^k(x), \lambda v) &= \bar{d}\Big((\prod_{0\leqslant m\leqslant k-1} \mu_m) g^k(x), \lambda v \Big) \\
&\leqslant \bar{d}\Big((\prod_{0\leqslant m\leqslant k-1} \mu_m) g^k(x), \lambda g^k(x) \Big) + \bar{d}(\lambda g^k(x), \lambda v) \\
&\leqslant (\sum_{m\geqslant k} |\log \mu_m|) + \bar{d}(g^k(x), v) \ ,
\end{aligned}$$

and combining (7.17) with (7.14), we get (7.10). □

8. Fixed points and eigenvectors of semidifferentiable nonexpansive maps over proper cones

As pointed out in Section 6, the map $f : \mathscr{C}_0([0,1]) \to \mathscr{C}_0([0,1])$ defined by $f(x)(t) = x(\frac{t}{2}) \vee (x(t) - 1)$ and which has $v \equiv 0$ as a unique fixed point, is such that $\mathrm{Id} - f'_v$ does not have Property (F) on $\mathscr{C}_0([0,1])$. One can show however that $\mathrm{Id} - f'_v$ has Property (F) on the Banach space $\mathscr{C}_0^\gamma([0,1])$ of Hölder continuous functions $x : [0,1] \to \mathbb{R}$ with exponent γ, such that $x(0) = 0$. Since the positive cone of $\mathscr{C}_0^\gamma([0,1])$ is not a normal cone, but only a proper cone, the results of Section 7

cannot be used. In order to treat this case, we thus need to prove a result similar to that of Section 7 but under the less restrictive condition that C is proper. The results of Section 6 cannot be applied. We are using rather Condition (A2) below, which will allow degree theory arguments (see [MPN02]).

Theorem 8.1. *Let C be a proper cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let $\psi \in C^* \setminus \{0\}$, and denote $\Sigma = \{x \in \operatorname{int} C \mid \psi(x) = 1\}$. Let G be a nonempty relatively open subset of Σ, and $g : G \to \Sigma$ be a nonexpansive map with respect to Hilbert's projective metric d. Assume that $v \in G$ is a fixed point of g. Make the following assumptions:*

(A1) *g is semi-differentiable at v;*
(A2) *There exists a relatively open neighborhood U of v in Σ, $U \subset G$, and a homogeneous generalized measure of noncompactness ν on X, such that $g|_U$ is a k-set contraction with $k < 1$ with respect to ν;*
(A3) *The fixed point of $g'_v : \psi^{-1}(0) \to \psi^{-1}(0)$ is unique: $x = g'_v(x)$, $x \in \psi^{-1}(0) \Rightarrow x = 0$.*

Then, v is the only fixed point of g in G: $g(w) = w$, $w \in G \implies w = v$.

Remark 8.2. In condition (A2), we could use any homogeneous, generalized measure of noncompactness ν, instead of the Kuratowski measure of noncompactness α.

The following lemma is the key result needed to prove Theorem 8.1.

Lemma 8.3. *Let assumptions and notations be as in Theorem 8.1. If $w \in \Sigma$, define, for $0 \leqslant t \leqslant 1$, $g_t : G \to \Sigma$ by $g_t(y) = (1-t)g(y) + tw$. Given $\varepsilon > 0$, there exists $\delta > 0$ such that for $0 \leqslant t \leqslant \delta$, there exists $y_t \in G$ with $\|y_t - v\| < \varepsilon$ and $g_t(y_t) = y_t$.*

Proof. Define $h_t(x) := g_t(v + x) - v$ for $x \in G_0 := -v + G \subset \psi^{-1}(0)$. By Assumption (A2) of Theorem 8.1, $h := h_0$ is a k-set contraction when restricted to the open neighborhood $-v + U$ of 0 in $\psi^{-1}(0)$. So $h'_0 = g'_v$, the semi-derivative of h at 0, is a k-set contraction on $(\psi^{-1}(0), \|\cdot\|)$ (this follows from (5.4a)). It follows that $(h_t)'_0 = (g_t)'_v = (1-t)g'_v$ is a k-set contraction on $(\psi^{-1}(0), \|\cdot\|)$.

We need to show that for $t > 0$, t small, $h_t(x) = x$ has a solution in $B_\varepsilon(0) := \{z \in \psi^{-1}(0) \mid \|z\| \leqslant \varepsilon\}$ (one may assume that ε is such that $B_\varepsilon(0) \subset -v + G$). For any fixed $t > 0$, we consider the homotopy:

$$x - (1-\lambda)h_t(x) - \lambda(h_t)'_0(x), \quad 0 \leqslant \lambda \leqslant 1, \quad x \in B_\varepsilon(0) \ .$$

Recall that, since g is semidifferentiable at v, with semiderivative g'_v:

$$g(v + x) = v + g'_v(x) + R(x)$$

where $\|R(x)\| \leqslant \eta(\|x\|)\|x\|$ and $\lim_{s \to 0^+} \eta(s) = 0$. It follows that

$$\begin{aligned} & x - (1-\lambda)h_t(x) - \lambda(h_t)'_0(x) \\ &= x - (1-\lambda)(g'_v(x) + R(x) + tw - tg(v+x)) - \lambda(1-t)g'_v(x) \\ &= x - g'_v(x) - (1-\lambda)R(x) - (1-\lambda)t(w - g(v+x)) + \lambda t g'_v(x) \ . \end{aligned}$$

For $\|x\| = 1$, $x \in \psi^{-1}(0)$, $x - g'_v(x) \neq 0$ by Assumption (A3). Because g'_v is a k-set contraction on $(\psi^{-1}(0), \|\cdot\|)$, it follows that there exists $c > 0$ such that

$$\|x - g'_v(x)\| \geqslant c \text{ for } \|x\| = 1, \ x \in \psi^{-1}(0) \ .$$

Indeed, by Lemma 5.2 $\mathrm{Id} - g'_v$ has Property (F), hence if, by way of contradiction, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of $\psi^{-1}(0)$ such that $\|x_n\| = 1$ and $\lim_{n \to \infty} x_n -$

$g'_v(x_n) = 0$, there exists a convergent subsequence. Thus, the limit satisfies $x - g'_v(x) = 0$ and $\|x\| = 1$, which contradicts Assumption (A3).

By the homogeneity of g'_v, it follows that

$$\|x - g'_v(x)\| \geqslant c\|x\|, \text{ for } x \in \psi^{-1}(0) \ .$$

Select $\varepsilon_1 \leqslant \varepsilon$, $\varepsilon_1 > 0$, so that $\eta(s) \leqslant c/2$ for $0 \leqslant s \leqslant \varepsilon_1$. It follows that

$$\|x - g'_v(x) - (1-\lambda)R(x)\| \geqslant \frac{c}{2}\|x\|, \text{ for } 0 \leqslant \|x\| \leqslant \varepsilon_1,\ x \in \psi^{-1}(0) \ .$$

Since $g(v+x)$ and $g'_v(x)$ are bounded in norm on $B_{\varepsilon_1}(0)$, it follows that there exists $\delta > 0$ so that

$$t\|w - g(v+x)\| + t\|g'_v(x)\| < \frac{c\varepsilon_1}{2}, \text{ for } \|x\| = \varepsilon_1,\ 0 \leqslant t \leqslant \delta \ .$$

Using this estimate, we see that, for $0 \leqslant t \leqslant \delta$, $\|x\| = \varepsilon_1$ and $0 \leqslant \lambda \leqslant 1$,

$$x - (1-\lambda)h_t(x) - \lambda(h_t)'_0(x) \neq 0 \ .$$

By the homotopy property for degree theory for k-set contractions

$$\deg(\mathrm{Id} - h_t, H, 0) = \deg(\mathrm{Id} - (h_t)'_0, H, 0) \ ,$$

where $H = \{z \in \psi^{-1}(0) \mid \|z\| < \varepsilon_1\}$.

Because g is nonexpansive with respect to d, by Lemma 4.2,(viii), we know that g'_v is nonexpansive on $\psi^{-1}(0)$, with respect to the seminorm ω_v defined by (2.7) (which is a norm on $\psi^{-1}(0)$). It follows that $x - \sigma(1-t)g'_v(x) \neq 0$ for $0 \leqslant \sigma \leqslant 1$ and $\|x\| = \varepsilon_1$, $x \in \psi^{-1}(0)$.

By the homotopy property of degree theory again,

$$\begin{aligned} \deg(\mathrm{Id} - (h_t)'_0, H, 0) &= \deg(\mathrm{Id} - \sigma(1-t)g'_v, H, 0) \qquad \text{for } 0 \leqslant \sigma \leqslant 1 \\ &= \deg(\mathrm{Id}, H, 0) = 1 \ . \end{aligned}$$

It follows that $\deg(\mathrm{Id} - h_t, H, 0) = 1$ for $0 \leqslant t \leqslant \delta$, so for $0 \leqslant t \leqslant \delta$, there exists $x_t \in H$ with $x_t = h_t(x_t)$. □

Proof of Theorem 8.1. Suppose, by way of contradiction, that there exists $w \in G$ with $w \neq v$ and $g(w) = w$. Select $r > 0$ so that $\|w - v\| > r$. Define $g_t : G \to \Sigma$ by $g_t(y) = (1-t)g(y) + tw$. Take $\delta > 0$ (by Lemma 8.3) so that, for $0 < t \leqslant \delta$, there exists $y_t \in G$ such that $\|y_t - v\| < r$ and $g_t(y_t) = y_t$. Note that $g_t(w) = w$ too, and we know (see [Nus88, Lemma 2.1, p. 45]) that

$$d(g_t(z_1), g_t(z_2)) < d(z_1, z_2)$$

if $0 < t \leqslant 1$ and $z_1, z_2 \in G$, $z_1 \neq z_2$. Taking $z_1 = y_t$, $z_2 = w$, $0 < t \leqslant \delta$, we obtain a contradiction. □

Theorem 8.4. *Let C be a proper cone with nonempty interior in a Banach space $(X, \|\cdot\|)$. Let $\psi \in C^* \setminus \{0\}$, and denote $\Sigma = \{x \in \operatorname{int} C \mid \psi(x) = 1\}$. Let G be a nonempty relatively open subset of Σ, and $g : G \to \Sigma$ be a nonexpansive map with respect to Hilbert's projective metric d. Assume that $v \in G$ is a fixed point of g. Make the following assumptions:*

(A1) *g is semi-differentiable at v;*

(A2) *There exists a relatively open neighborhood U of v in Σ, $U \subset G$, and an integer $n \geqslant 1$ such that $g^n|_U$ is a k-set contraction with $k < 1$. If $n > 1$, assume also that g is uniformly continuous on some relatively open neighborhood V of v in Σ, $V \subset U$.*

(A3) *The fixed point of* $(g'_v)^n : \psi^{-1}(0) \to \psi^{-1}(0)$ *is unique:* $x = (g'_v)^n(x),\ x \in \psi^{-1}(0) \Rightarrow x = 0$.

Then, v is the only fixed point of g^n in G (and hence the only fixed point of g in G): $g^n(w) = w,\ w \in G \implies w = v$.

Proof. The map g^n is defined on some relatively open subset G_n of Σ (take $G_n = G \cap g^{-1}(G) \cap \cdots \cap g^{-(n-1)}(G)$), and it is semidifferentiable at v, with $(g^n)'_v = (g'_v)^n$ by the chain rule. Applying Theorem 8.1 to g^n, we get the conclusion of Theorem 8.4. □

9. Eigenvectors of differentiable nonexpansive maps on cones

In this section, we specialize the previous results to the case where f is differentiable at the fixed point, and compare the results obtained in this way with the ones of [Nus88].

Corollary 9.1 (Uniqueness of eigenvectors in Σ, differentiable case)**.** *Let C be a normal cone with nonempty interior in a Banach space $(X, \|\cdot\|)$, let $\psi \in C^* \setminus \{0\}$, and denote $\Sigma = \{x \in \operatorname{int} C \mid \psi(x) = 1\}$. Let G be an open subset of C and $f : G \to \operatorname{int} C$ be a map such that $f|_{G\cap\Sigma}$ is nonexpansive with respect to Hilbert's projective metric d. Assume that $v \in G \cap \Sigma$ is a fixed point of f: $f(v) = v$. Make the following assumptions:*

(A1) *f is differentiable at v;*
(A2) *The linear operator $(\mathrm{Id} - f'_v)|_{\psi^{-1}(0)} : \psi^{-1}(0) \to X$ is semi-Fredholm with index in $\mathbb{Z} \cup \{-\infty\}$;*
(A3) *The fixed point of f'_v in $\psi^{-1}(0)$ is unique: $f'_v(x) = x,\ x \in \psi^{-1}(0) \Rightarrow x = 0$;*
(A4) *$f'_v(C) \subset C$;*
(A5) *$f'_v(v) \leqslant v$.*

Then, the eigenvector of f in $G \cap \Sigma$ is unique: $\exists \lambda > 0,\ f(w) = \lambda w,\ w \in G \cap \Sigma \Rightarrow w = v$.

Proof. We only need to verify that f satisfies Assumptions (A1)–(A5) of Theorem 7.5. Clearly, Assumption (A1) of Corollary 9.1 implies Assumption (A1) of Theorem 7.5. Under this condition, f'_v is linear, hence, Assumption (A2) of Theorem 7.5 is equivalent to Assumption (A2) of Corollary 9.1 (see [Hör94, Proposition 19.1.3] or [Kat95, Chapter IV, Theorems 5.10 and 5.11]). Assumption (A3) of Theorem 7.5 is identical to Assumption (A3) of Corollary 9.1. Since f'_v is linear, f'_v is order preserving if, and only if, $f'_v(x) \geqslant 0$ when $x \geqslant 0$, that is $f'_v(C) \subset C$, which shows that Assumption (A4) of Theorem 7.5 is equivalent to Assumption (A4) of Corollary 9.1 when f'_v is linear. Also, since f'_v is linear, Lemma 4.3 shows that Assumption (A5) of Corollary 9.1 implies Assumption (A5) of Theorem 7.5. □

Remark 9.2. The assumptions of Corollary 9.1 arise when specializing the ones of Theorem 7.5 to the case in which f'_v is linear. However Assumption (A2) can be replaced by the apparently more restrictive assumption that $\mathrm{Id} - f'_v$ is Fredholm of index 0, or that $(\mathrm{Id} - f'_v)|_{\psi^{-1}(0)}$ is Fredholm of index -1, without loss of generality.

Indeed, under the other assumptions, in particular when (A1), (A4) and (A5) hold, and when v is in the interior of C, we get that f'_v is linear continuous and satisfies $\tilde{r}_C(f'_v) \leqslant 1$. Then for all $0 < t < 1$, $\mathrm{Id} - t f'_v$ is one-one, and thus Fredholm of index 0.

Moreover, since $\ker\psi$ is closed and of codimension one, it follows from standard properties of Fredholm operators (see [Hör94, § 19.1]) that $\mathrm{Id}-f'_v : X \to X$ is semi-Fredholm if and only if $(\mathrm{Id}-f'_v)|_{\psi^{-1}(0)} : \psi^{-1}(0) \to X$ is semi-Fredholm, and the index of $\mathrm{Id}-f'_v$ is equal to the one of $(\mathrm{Id}-f'_v)|_{\psi^{-1}(0)}$ plus 1. This implies that under Assumption (A2), $\mathrm{Id}-f'_v$ is semi-Fredholm.

From the latter property, the fact that all operators $\mathrm{Id}-tf'_v$ with $t<1$ are Fredholm of index 0, and the continuity of the Fredholm index on the set of semi-Fredholm operators, we deduce that $\mathrm{Id}-f'_v$ is Fredholm of index 0, or equivalently that $(\mathrm{Id}-f'_v)|_{\psi^{-1}(0)}$ is Fredholm of index -1. Conversely, if $(\mathrm{Id}-f'_v)|_{\psi^{-1}(0)}$ is Fredholm of index -1, then (A2) holds trivially.

We shall give now a corollary of Corollary 9.1, which extends partially [Nus88, Theorem 2.5]. For a bounded linear operator $L : X \to X$ on a Banach space $(X,\|\cdot\|)$, we denote by $r(L)$ the spectral radius of L, which coincides with the Bonsall spectral radius $\tilde r_X(L)$ in (5.1). We denote by $N(L)$ the null space of L. Let C be a proper cone of X. We say that L satisfies the *weak Krein-Rutman* (WKR) condition with respect to C if $L(C)\subset C$, and either $r(L)=0$, or $r:=r(L)>0$ and (i) there exist $u\in C\setminus\{0\}$ such that $N(r\,\mathrm{Id}-L)=\{\lambda u \mid \lambda\in\mathbb{R}\}$, and (ii) $r\,\mathrm{Id}-L$ is a semi-Fredholm operator. Condition (WKR) is easier to check than the Krein-Rutman (KR) condition used in [Nus88, Definition 2.1], where the condition (ii) is replaced by the condition that $r\,\mathrm{Id}-L$ is a Fredholm operator with index 0, and where it is also assumed that, when $r>0$, L^* has an eigenvector $u^*\in C^*\setminus\{0\}$ with eigenvalue r such that $u^*(u)>0$. However, if (WKR) holds, then, by the same arguments as in Remark 9.2, if $r:=r(L)>0$, then, for all $0<t<1$, $r\mathrm{Id}-tL$, and so $r\mathrm{Id}-L$, are necessarily Fredholm of index 0, as requested by the (KR) condition. Furthermore, a refinement of a theorem of Krein and Rutman (see [SW99]) implies that L^* has an eigenvector u^* in C^* with eigenvalue r. Thus, the only difference in generality between condition (WKR) and condition (KR) is the requirement that $u^*(u)>0$. Even if a variety of conditions imply that $u^*(u)>0$, it may happen that $u^*(u)=0$.

Corollary 9.3 (Uniqueness of eigenvectors, differentiable case)**.** *Let C be a normal cone with nonempty interior in a Banach space $(X,\|\cdot\|)$, let $\psi\in C^*\setminus\{0\}$, and denote $\Sigma=\{x\in\operatorname{int}C \mid \psi(x)=1\}$. Let G be an open subset of $(X,\|\cdot\|)$ included in C and $f:G\to\operatorname{int}C$ be a map such that $f|_{G\cap\Sigma}$ is nonexpansive with respect to Hilbert's projective metric d. Assume that $v\in G\cap\Sigma$ is a fixed point of f: $f(v)=v$. Make the following assumptions:*

- (A1) *f is differentiable at v;*
- (A2) *The linear operator f'_v satisfies the* WKR *condition with respect to C;*
- (A3) *If $u\in C\setminus\{0\}$ is an eigenvector of f'_v with eigenvalue $r(f'_v)>0$, then $\psi(u)>0$;*
- (A4) *There exist $\delta>0$ such that $\delta\leqslant 1$ and $tf(v)\leqslant f(tv)$ for all $1-\delta\leqslant t\leqslant 1$.*

Then, the eigenvector of f in $G\cap\Sigma$ is unique: $\exists\lambda>0,\ f(w)=\lambda w,\ w\in G\cap\Sigma\Rightarrow w=v$.

Proof. We only need to verify that f satisfies Assumptions (A1)–(A5) of Corollary 9.1. Assumption (A1) of Corollary 9.3 corresponds to (A1) of Corollary 9.1. Assumption (A2) of Corollary 9.3 implies that $f'_v(C)\subset C$, by definition of the WKR condition, that is Assumption (A4) of Corollary 9.1. By Lemma 4.2, (vi), we get that Assumption (A4) of Corollary 9.3 implies Assumption (A5) of Corollary 9.1,

since f'_v is linear. It remains to show Asssumptions (A2) and (A3) of Corollary 9.1. From Assumptions (A4) and (A5) of Corollary 9.1, f'_v is homogeneous, order preserving and satisfies $\mathrm{cw}_C(f'_v) \leqslant 1$, where for any homogeneous map $h : C \to C$, $\mathrm{cw}_C(h) := \inf\{\lambda > 0 \mid \exists x \in \operatorname{int} C,\ h(x) \leqslant \lambda x\}$. From [AGN11, Lemma 7.2], it follows that $r_C(f'_v) \leqslant 1$ and since f'_v is linear, we get that $r(f'_v) \leqslant 1$. Consider first the case where $r(f'_v) < 1$. Then, by Proposition 5.3, $\mathrm{Id} - f'_v$ has Property (F), or is a semi-Fredholm operator with index in $\mathbb{Z} \cup \{-\infty\}$, which shows Assumption (A2) of Corollary 9.1, and $N(\mathrm{Id} - f'_v) = \{0\}$, which implies Assumption (A3) of Corollary 9.1. Moreover, by Proposition 5.3, $\mathrm{Id} - f'_v$ has Property (F), or is a semi-Fredholm operator with index in $\mathbb{Z} \cup \{-\infty\}$, which shows Assumption (A2) of Corollary 9.1. Consider now the case where $r(f'_v) = 1$. Then, by Assumption (A2) of Corollary 9.3, $\mathrm{Id} - f'_v$ is a semi-Fredholm operator, and since the dimension of $N(\mathrm{Id} - f'_v)$ is finite (equal to 1), the index of $\mathrm{Id} - f'_v$ is in $\mathbb{Z} \cup \{-\infty\}$, hence Assumption (A2) of Corollary 9.1 holds. Let x be a fixed point of f'_v in $\psi^{-1}(0)$, that is $x \in N(\mathrm{Id} - f'_v) \cap \psi^{-1}(0)$. By By Assumption (A3) of Corollary 9.3, we deduce that $x = 0$. This shows that Assumption (A3) of Corollary 9.1 holds. □

Corollary 9.3 extends partially [Nus88, Theorem 2.5]. We obtain the same conclusion with different assumptions: the condition WKR is replaced by the stronger KR condition; the map f is assumed to be $\mathscr{C}_1$, whereas we only assume f to be differentiable at v; Condition (A4) is assumed for all $v \in \Sigma \cap G$, whereas we only require it for the fixed point v; but we require the cone C to be normal, whereas the result of [Nus88] is valid for a general proper cone C.

10. Application to nonexpansive self-maps of AM-spaces with unit

In this section, we give additive versions of the results of Section 7, motivated by the case of Shapley operators of zero-sum games with a compact state space K. The latter operators are order preserving and sup-norm nonexpansive maps acting on a Banach space $\mathscr{C}(K)$.

10.1. **Uniqueness of the fixed point and of the additive eigenvector.** Since for any Banach space $(X, \|\cdot\|)$, the corresponding metric space (V, d) (with $V = X$ and $d(x, y) = \|x - y\|$) satisfies Assumptions (B1), (B2) and (B3), the following additive versions of Theorems 7.1 and 7.2 are obtained directly from Theorems 6.1 and 6.8 together with Remark 6.9.

Corollary 10.1. *Let $(X, \|\cdot\|)$ be an AM-space with unit, let G be a nonempty open subset of X and let $F : G \to X$ be a nonexpansive map with respect to $\|\cdot\|$. Let $v \in G$ be a fixed point of F: $F(v) = v$. Make the following assumptions:*

(A1) *F is semidifferentiable at v;*
(A2) *The map $\mathrm{Id} - F'_v : X \to X$ has Property* (F)*;*
(A3) *The fixed point of $F'_v : X \to X$ is unique: $F'_v(x) = x$, $x \in X \Rightarrow x = 0$.*

Then, the fixed point of F in G is unique: $F(w) = w$, $w \in G \Rightarrow w = v$.

Corollary 10.2. *Let $(X, \|\cdot\|)$ be an AM-space with unit, let G be a non-empty open subset of X and let $F : G \to G$ be a nonexpansive map with respect to $\|\cdot\|$. If F has a fixed point $v \in G$ and if F is semidifferentiable at v, with $\tilde{r}(F'_v) < 1$, the fixed point of F in G is unique. Moreover, if G is connected, we have:*

$$\forall x \in G,\ \limsup_{k\to\infty} \|F^k(x) - v\|^{1/k} \leqslant \tilde{r}(F'_v) \ . \tag{10.1}$$

Applying Theorem 6.1 to the metric spaces (V,d) corresponding to the Banach spaces $(\psi^{-1}(0),\|\cdot\|)$ and $(\psi^{-1}(0),\omega)$ with $\psi\in(X^+)^*\setminus\{0\}$, or replacing X by $\psi^{-1}(0)$ and perhaps $\|\cdot\|$ by ω in Corollary 10.1, one would have obtained an additive version of Theorem 7.3. The following additive version of Theorem 7.5, will be derived from Corollary 10.1.

Corollary 10.3. *Let $(X,\|\cdot\|)$ be an AM-space with unit, denoted by e, and let $\psi\in(X^+)^*\setminus\{0\}$. Let G be an open subset of X and let $F:G\to X$ be a map such that $F|_{G\cap\psi^{-1}(0)}$ is nonexpansive with respect to ω. Let $v\in G\cap\psi^{-1}(0)$ be a fixed point of F: $F(v)=v$. Make the following assumptions:*

(A1) *F is semidifferentiable at v;*
(A2) *The map $(\mathrm{Id}-F'_v)|_{\psi^{-1}(0)}:\psi^{-1}(0)\to X$ has Property* (F)*;*
(A3) *The fixed point of F'_v in $\psi^{-1}(0)$ is unique: $F'_v(x)=x$, $x\in\psi^{-1}(0)\Rightarrow x=0$;*
(A4) *F'_v is order preserving;*
(A5) *F'_v is additively subhomogeneous.*

Then, the additive eigenvector of F in $G\cap\psi^{-1}(0)$ is unique: $\exists\lambda\in\mathbb{R},\ F(w)=\lambda e+w$, $w\in G\cap\psi^{-1}(0)\Rightarrow w=v$.

Proof. Since the assumptions and conclusions of the corollary depend only on $\psi^{-1}(0)$, we may assume that $\psi(e)=1$. Let $\tilde F:G\to\psi^{-1}(0)$, $x\mapsto F(x)-\psi(F(x))e$ and denote $H=\tilde F|_{G\cap\psi^{-1}(0)}$. We shall prove that H satisfies the assumptions of Corollary 10.1 for the Banach space $(\psi^{-1}(0),\omega)$. Since $F|_{G\cap\psi^{-1}(0)}$ is nonexpansive with respect to ω, so is H. Since $F(v)=v$ and $v\in G\cap\psi^{-1}(0)$, $H(v)=v$. Since F is semidifferentiable at v and the map $R:X\to\psi^{-1}(0)$, $x\mapsto x-\psi(x)e$ is linear, thus differentiable at any point, $\tilde F=R\circ F$ is semidifferentiable at v, by Lemma 3.4, and $\tilde F'_v=R\circ F'_v$. This implies that H is semidifferentiable at v with $H'_v=\tilde F'_v|_{\psi^{-1}(0)}:\psi^{-1}(0)\to\psi^{-1}(0)$. We thus get Assumption (A1) of Corollary 10.1. Taking $C=X^+$, and using Assumptions (A4) and (A5) of the Corollary 10.3, we obtain by the same arguments as in the proof of Theorem 7.5, the following inequality:

$$\|x-F'_v(x)\|\leqslant 2\,\|x-H'_v(x)\|\quad\forall x\in\psi^{-1}(0)\ . \tag{10.2}$$

Then, using $H'_v(x)=F'_v(x)-\psi(F'_v(x))e$ and (10.2), we obtain, by the same arguments as in the proof of Theorem 7.5, Assumptions (A2) and (A3) of Corollary 10.1 with X replaced by $\psi^{-1}(0)$. If $w\in G\cap\psi^{-1}(0)$ satisfies $F(w)=\lambda e+w$ for some $\lambda\in\mathbb{R}$, we get $H(w)=w$, and by Corollary 10.1, we obtain $w=v$. □

Remark 10.4. As for Theorem 7.5, Corollary 10.3 can be applied in the following situations. From Lemma 4.7, Assumption (A4) of Corollary 10.3 is fulfilled as soon as F is order preserving in a neighborhood of v, and Assumption (A5) is fulfilled as soon as F is additively subhomogeneous in a neighborhood of v. Moreover, these properties imply, by Lemma 4.6, that $F|_{G\cap\psi^{-1}(0)}$ is nonexpansive with respect to ω.

The following additive version of Corollary 7.7 will be derived from Corollary 10.3.

Corollary 10.5. *Let $(X,\|\cdot\|)$ be an AM-space with unit, denoted by e, and let $F:X\to X$ be an additively homogeneous and order preserving map. Assume that $S=\{x\in X\mid F(x)=x\}$ is nonempty and that $v\in S$. Make the following assumptions:*

(A1) *F is semidifferentiable at v;*
(A2) *The map $\mathrm{Id} - F'_v : X \to X$ has Property (F);*
(A3) *if $F'_v(x) = x$ for some $x \in X$, then $x \in \{\lambda e \mid \lambda \in \mathbb{R}\}$;*

Then, $S = \{v + \lambda e \mid \lambda \in \mathbb{R}\}$.

Proof. Let us check that F satisfies the assumptions of Corollary 10.3. Consider $G = X$, $\psi \in (X^+)^* \setminus \{0\}$ such that $\psi(e) = 1$. Since F is additively homogeneous and $v \in S$, $v + \lambda e \in S$ for all $\lambda \in \mathbb{R}$. Moreover, since F is semidifferentiable at v, F is semidifferentiable at $v + \lambda e$ for all $\lambda \in \mathbb{R}$, with $F'_{v+\lambda e} = F'_v$. Taking $\lambda = -\psi(v)$, we get $\psi(v+\lambda e) = 0$ and $v+\lambda e$ satisfies all the assumptions of the corollary. Hence, replacing v by $v + \lambda e$, we may assume that $\psi(v) = 0$.

We have : $v \in G \cap \psi^{-1}(0)$ and $F(v) = v$. Since F is order preserving and additively homogeneous, $F|_{\psi^{-1}(0)}$ is nonexpansive with respect to ω (see Lemma 4.6). Assumptions (A1) and (A2) of Corollary 10.3 are implied by Assumptions (A1) and (A2) of Corollary 10.5. Corollary 10.3 is deduced from Lemma 4.7, (i) and (iii), using the fact that F is order preserving and additively homogeneous. This completes the proof of the assumptions of Corollary 10.3.

Let $x \in S$, and denote $\lambda = \psi(x)$. Since F is additively homogeneous, $y = x - \lambda e$ satisfies $F(y) = y$ and $y \in \psi^{-1}(0)$. From Corollary 10.3, this implies $y = v$, hence $x = \lambda e + v$. Since we already proved above the converse implication, this yields the conclusion of the corollary. □

The following is the additive version of Theorem 7.8.

Corollary 10.6. *Let $(X, \|\cdot\|)$ be an AM-space with unit, denoted by e, and let $F : X \to X$ be an additively homogeneous and order preserving map. Assume that $S = \{x \in X \mid F(x) = x\}$ is nonempty and that $v \in S$. Make the following assumptions:*

(A1) *F is semidifferentiable at v;*
(A2) *$\tilde{r}(F'_v) < 1$, where $\tilde{r}$ is defined with respect to the seminorm ω_e, as in (7.7), (7.8) (with $C = X$).*

Then, for all $x \in X$,

$$\limsup_{k\to\infty} \omega_e(F^k(x) - v)^{1/k} \leqslant \tilde{r}(F'_v) \ , \tag{10.3}$$

and there is a scalar λ (depending on x), such that

$$\limsup_{k\to\infty} \|F^k(x) - \lambda e - v\|^{1/k} \leqslant \tilde{r}(F'_v) \ . \tag{10.4}$$

Proof. We define H and ψ as in the proof of Corollary 10.3, so that H leaves invariant the Banach space $\psi^{-1}(0)$ equipped with the norm ω_e. Then, Corollary 10.2 implies that

$$\limsup_{k\to\infty} \omega_e(F^k(x) - v)^{1/k} = \limsup_{k\to\infty} \omega_e(H^k(x) - v)^{1/k} \leqslant \tilde{r}(H'_v) = \tilde{r}(F'_v) \ ,$$

which shows (10.3). Now, we use the additive analogue of the 1-cocycle formula (7.15), namely

$$F^k(x) = \big(\psi(F \circ H^{k-1}(x)) + \cdots + \psi(F \circ H^0(x))\big)e + H^k(x)$$

and by a straightforward adaptation of the argument of the second part of the proof of Theorem 7.8, we conclude that (10.4) holds for some scalar $\lambda \in \mathbb{R}$. □

Remark 10.7. One may have derived Corollary 10.5 from Corollary 7.7 by using the Kakutani-Krein theorem and exp-log transformations as follows (see Section 2.3). Let $\imath : X \to \mathscr{C}(K)$ (where K is a compact set) $C = \mathscr{C}^+(K)$, $\operatorname{int} C$ and $\log : \operatorname{int} C \to \mathscr{C}(K)$ be defined as in Section 2.3. Denote $\jmath = \imath^{-1} \circ \log : \operatorname{int} C \to X$. Then, $\jmath^{-1} = \exp \circ \imath$, where $\exp = \log^{-1}$. To a map $F : X \to X$, one associate the map $f : \operatorname{int} C \to \operatorname{int} C$, $f = \jmath^{-1} \circ F \circ \jmath$. Any additive property (homogeneity, subhomogeneity, order preserving property) of F is transformed into its multiplicative version for f. Moreover, since $\imath$ is an isometry, F is Lipschitz continuous for the norm of X if, and only if, f is Lipschitz continuous for the Thompson metric of C and this implies that f is locally Lipschitz continuous for the sup-norm of $\mathscr{C}(K)$. Hence, the differentiability of the exp, log, $\imath$ and $\imath^{-1}$ transformations, implies, from Lemma 3.4, that, when F is Lipschitz continuous, the semidifferentiability of f is equivalent to that of F. This allows us to derive Corollary 10.5 from Corollary 7.7. However, in order to derive Corollary 10.3 from Theorem 7.5, and Corollary 10.6 from Theorem 7.8 one should have generalized first Theorem 7.5 and Theorem 7.8 to the case where ψ is a nonlinear order preserving homogeneous map from $\operatorname{int} C$ to $\mathbb{R}^+$ (the linearity of ψ is not preserved by taking “log-glasses”).

10.2. **An example of stochastic game.** As a simple illustration of the present results, consider the following zero-sum two player game, which may be thought of as a variant (with additive rewards) of the Richman games [LLP+99] or of the stochastic tug-of-war games [PSSW09] arising in the discretization of the infinity Laplacian [Obe05].

Let $G = (V, E)$ denote a (finite) directed graph with set of nodes V and set of arcs $E \subset V \times V$. Loops, i.e., arcs of the form (i, i) are allowed. We assume that every node has at least one successor (for every $i \in V$, there is at least one $j \in V$ such that $(i, j) \in E$). We associate to every arc $(i, j) \in E$ a payment $A_{ij} \in \mathbb{R}$. Two players, called “Max” and “Min”, will move a token on this digraph, tossing an unbiased coin at each turn, to decide who plays the turn. The player (Max or Min) who just won the right to play the turn must choose a successor node j (so that $(i, j) \in E$) and move the token to this node, Then, Player Max receives the payment A_{ij} from Player Min. We denote by $v_i(k)$ the *value* of this game in k turns, provided the initial state is $i \in V$. We refer the reader to [MS96, FV97, NS03] for background on zero-sum games, including the definition and properties of the value. In particular, standard dynamic programming arguments show that the value of the game in k turns does exist, and that the *value vector* $v(k) := (v_i(k)) \in \mathbb{R}^V$ satisfies

$$v(k) = F(v(k-1)), \qquad v(0) = 0.$$

where F (the Shapley operator) is the map $\mathbb{R}^V \to \mathbb{R}^V$ given by

$$F_i(x) = \frac{1}{2}\Big(\max_{j \in N\ (i,j) \in E} (A_{ij} + x_j) + \min_{j \in N\ (i,j) \in E} (A_{ij} + x_j) \Big), \qquad \forall i \in V \ . \tag{10.5}$$

We are interested in the *mean payoff* vector

$$\chi(F) := \lim_{k \to \infty} v(k)/k = \lim_{k \to \infty} F^k(0)/k;$$

hence, $\chi_i(F)$ represents the mean payoff per time unit starting from the initial state i, when the number of turns k tends to infinity. We shall consider, for simplicity, the case in which F has an additive eigenvector $u \in \mathbb{R}^n$ with associated eigenvalue

$\mu \in \mathbb{R}$, meaning that $F(u) = u + \mu e$ where e denotes the unit vector of $\mathbb{R}^n$. Then,

$$\chi(F) = \mu e \ .$$

Actually, the generalized Perron-Frobenius theorem in [GG04] implies that the additive eigenpair (u, μ) does exist if the graph of the game G is strongly connected.

The vector u, which is sometimes called *bias*, or *potential* in the dynamic programming literature, can be interpreted as an invariant terminal payoff. Indeed, consider the auxiliary game in which all the rewards A_{ij} are replaced by $A_{ij} - \mu$, and a terminal payment u_i is paid by Min to Max if the terminal state is i. Then, the equation $F(u) = \mu e + u$, or $-\mu e + F(u) = u$ means that the value of this modified game is independent of the number of turns (the operator $x \mapsto -\mu e + F(x)$ being interpreted as the dynamic programming operator of this modified game). An interest of a bias vector is that it determines stationary optimal strategies for both players, by selecting the actions attaining the maximum and minimum in the expression of $F(u)$.

The map F is semidifferentiable. To see this, let $E_i^+(x)$ and $E_i^-(x)$ denote the set of nodes j attaining the maximum and the minimum in (10.5). An application of Theorem 3.8 shows that the semidifferential of F at point u does exist and is given by

$$F'_u(x) := \frac{1}{2}\big(\max_{j \in E_i^+(u)} x_j + \min_{j \in E_i^-(u)} x_j \big).$$

Consider now as an example the digraph of Figure 1. The corresponding Shapley operator is given by

$$F(x) = \begin{pmatrix} \frac{1}{2}\big(\max(3+x_1, 4+x_2, x_3) + \min(3+x_1, 4+x_2, x_3)\big) \\ \frac{1}{2}\big(\max(x_1, 3+x_2, -7+x_3) + \min(x_1, 3+x_2, -7+x_3)\big) \\ \frac{1}{2}\big(\max(3+x_1, 2+x_2) + \min(3+x_1, 2+x_2)\big) \end{pmatrix} \ .$$

The vector $u = (5, 0, 4)$ can be checked to be an additive eigenvector of F, with additive eigenvalue $\mu = 1$, i.e., $F(u) = \mu e + u$, so that the mean payoff per time unit is equal to 1 for all initial states (Max is winning 1 per time unit). We get

$$F'_u(x) = \begin{pmatrix} \frac{1}{2}\big(x_1 + \min(x_2, x_3)\big) \\ \frac{1}{2}\big(x_1 + x_3\big) \\ \frac{1}{2}\big(x_1 + x_2\big) \end{pmatrix} \ .$$

Let $\mathsf{t}(x) := \max_{1 \leqslant i \leqslant n} x_i$, $\mathsf{b}(x) := \min_{1 \leqslant i \leqslant n} x_i$, and $\omega(x) := \omega_e(x) = \mathsf{t}(x) - \mathsf{b}(x)$. One readily checks that

$$\omega(F'_u(x)) = \frac{1}{2}\big(x_1 + \max(x_2, x_3)\big) - \frac{1}{2}\big(x_1 + \min(x_2, x_3)\big) \leqslant \frac{1}{2}\omega(x)$$

Hence, F'_u is a contraction of rate $1/2$ in the seminorm ω, which implies that $\tilde{r}(F'_u) \leqslant 1/2$. In particular that every fixed point x of F'_u satisfies $x_1 = x_2 = x_3$. Hence, Corollary 10.5 shows that the set of additive eigenvectors of F is precisely $S = \{u + \lambda e \mid \lambda \in \mathbb{R}\}$ (in other words, the bias vector is unique up to an additive constant). Moreover, Corollary 10.6 implies that for all $x \in \mathbb{R}^3$,

$$\limsup_{k \to \infty} \omega(F^k(x) - u)^{1/k} \leqslant 1/2 \ ,$$

and that there is a constant $\lambda \in \mathbb{R}$, depending on x, such that

$$\limsup_{k \to \infty} \|F^k(x) - \lambda e - u\|^{1/k} \leqslant 1/2 \ .$$

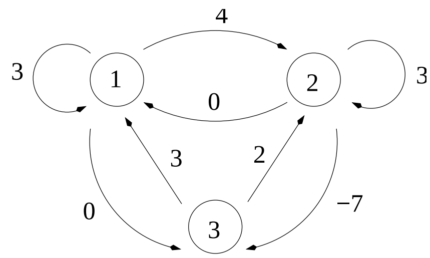

FIGURE 1. An additive version of Richman games

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Marianne Akian, INRIA and CMAP. Address: CMAP, École Polytechnique, 91128 Palaiseau Cedex, France

E-mail address: marianne.akian@inria.fr

Stéphane Gaubert, INRIA and CMAP. Address: CMAP, École Polytechnique, 91128 Palaiseau Cedex, France

E-mail address: stephane.gaubert@inria.fr

Roger Nussbaum, Mathematics Department, Hill Center, Rutgers University, 110 Frelinghuysen Road, Piscataway, New Jersey, U.S.A. 08854-8019

E-mail address: nussbaum@math.rutgers.edu